HAUSDORFF DIMENSION OF THE SET OF SINGULAR PAIRS

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Abstract. In this paper we show that the Hausdorff dimension of the set of singular pairs is $\frac{4}{3}$. We also show that the action of $\operatorname{diag}(e^t, e^t, e^{-2t})$ on $\mathrm{SL}_3\,\mathbb{R}/\,\mathrm{SL}_3\,\mathbb{Z}$ admits divergent trajectories that exit to infinity at arbitrarily slow prescribed rates, answering a question of A.N. Starkov. As a by-product of the analysis, we obtain a higher dimensional generalisation of the basic inequalities satisfied by convergents of continued fractions. As an illustration of the techniques used to compute Hausdorff dimension, we show that the set of real numbers with divergent partial quotients has Hausdorff dimension $\frac{1}{2}$.

1. Introduction

Let $\mathrm{Sing}(d)$ denote the set of all singular vectors in $\mathbb{R}^d$. Recall that $\mathbf{x} \in \mathbb{R}^d$ is said to be *singular* if for every $\delta > 0$ there exists T_0 such that for all $T > T_0$ the system of inequalities

$$\|q\mathbf{x} - \mathbf{p}\| < \frac{\delta}{T^{1/d}} \quad \text{and} \quad 0 < q < T \tag{1}$$

admits an integer solution $(\mathbf{p}, q) \in \mathbb{Z}^{d+1}$. Since $\mathrm{Sing}(d)$ contains every rational hyperplane in $\mathbb{R}^d$, its Hausdorff dimension is between $d-1$ and d. In this paper, we prove

Theorem 1.1. *The Hausdorff dimension of* $\mathrm{Sing}(2)$ *is* $\frac{4}{3}$.

Singular vectors that lie on a rational hyperplane are said to be *degenerate*. Implicit in this terminology is the expectation that the set $\mathrm{Sing}^*(d)$ of all *nondegenerate* singular vectors is somehow larger than the union of all rational hyperplanes in $\mathbb{R}^d$, which is a set of Hausdorff dimension $d-1$. The papers [1], [24], [19] and [2] give lower bounds on

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certain subsets of $\mathrm{Sing}^*(d)$ that, in particular, imply $\mathrm{H.dim}\,\mathrm{Sing}^*(d) \geq d-1$. Theorem 1.1 shows that strict inequality holds in the case $d = 2$.

Divergent trajectories. There is a well-known dynamical interpretation of singular vectors. Let G/Γ be the space of oriented unimodular lattices in $\mathbb{R}^{d+1}$, where $G = \mathrm{SL}_{d+1}\mathbb{R}$ and $\Gamma = \mathrm{SL}_{d+1}\mathbb{Z}$. A path $(\wedge_t)_{t\geq 0}$ in G/Γ is said to be *divergent* if for every compact subset $K \subset G/\Gamma$ there is a time T such that $\wedge_t \notin K$ for all $t > T$. By Mahler's criterion, $(\wedge_t)$ is divergent iff the length of the shortest nonzero vector of $\wedge_t$ tends to zero as $t \to \infty$. It is not hard to see that $\mathbf{x} \in \mathbb{R}^d$ is singular iff $\ell(g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}) \to 0$ as $t \to \infty$, where

$$g_t = \begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix}, \quad h_{\mathbf{x}} = \begin{pmatrix} 1 & & & -x_1 \\ & \ddots & & \vdots \\ & & 1 & -x_d \\ & & & 1 \end{pmatrix},$$

and $\ell(\cdot)$ denotes the length of the shortest nonzero vector. Thus, $\mathbf{x}$ is singular if and only if $(g_t h_{\mathbf{x}}\Gamma)_{t\geq 0}$ is a divergent trajectory of the homogeneous flow on G/Γ induced by the one-parameter subgroup (g_t) acting by left multiplication.

As a corollary of Theorem 1.1 we have

Corollary 1.2. *Let $D \subset \mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ be the set of points that lie on divergent trajectories of the flow induced by (g_t). Then* $\mathrm{H.dim}\, D = 7\frac{1}{3}$.

Proof. Let $P := \{p \in G | g_t p g_{-t}$ stays bounded as $t \to \infty\}$ and note that every $g \in G$ can be written as $p h_{\mathbf{x}} \gamma$ for some $p \in P$, $\mathbf{x} \in \mathbb{R}^d$ and $\gamma \in \Gamma$. Since the distance between $g_t p h$ and $g_t h$ with respect to any right-invariant metric on G stays bounded as $t \to \infty$, it follows that

$$D = \cup_{\mathbf{x}\in \mathrm{Sing}(d)} P h_{\mathbf{x}} \Gamma.$$

Since P is a manifold and Γ is countable, the Hausdorff codimension of D in G/Γ coincides with that of $\mathrm{Sing}(d)$ in $\mathbb{R}^d$. □

Further results about singular vectors and divergent trajectories can be found in the papers [10], [11], [22] and [23].

Related results. It should be mentioned that the notion of a singular vector is dual to that of a badly approximable vector. Recall that $\mathbf{x} \in \mathbb{R}^d$ is *badly approximable* if there is a $c > 0$ such that $\|q\mathbf{x} - \mathbf{p}\| > cq^{-1/d}$ for all $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$. As with $\mathrm{Sing}(d)$, the set $\mathrm{BA}(d)$ of badly approximable vectors in $\mathbb{R}^d$ admits a characterisation in terms of flow on G/Γ induced by (g_t). Specifically, $\mathbf{x} \in \mathrm{BA}(d)$ if and only if $(g_t h_{\mathbf{x}}\Gamma)_{t\geq 0}$ is a *bounded trajectory*, i.e. its closure in G/Γ is compact. In

[20] Schmidt showed that $\mathrm{H.dim}\,\mathrm{BA}(d) = d$, which implies that the set $B \subset G/\Gamma$ of points that lie on bounded trajectories of the flow induced by (g_t) has $\mathrm{H.dim}\, B = \dim G$. In [9] Kleinbock-Margulis generalised the latter statement to the setting of non-quasi-unipotent homogeneous flows where G is a connected real semisimple Lie group, Γ a lattice in G, and (g_t) a one-parameter subgroup such that $\mathrm{Ad}\, g_1$ has an eigenvalue of absolute value $\neq 1$.

For divergent trajectories of such flows, Dani showed in [5] that if Γ is of "rank one" then the set $D \subset G/\Gamma$ of points that lie on divergent trajectories of the flow is a countable union of proper submanifolds, implying that its Hausdorff dimension is integral and strictly less than $\dim G$. Much less is known in the case of lattices of higher rank. In the special case of the reducible lattice $\Gamma = (\mathrm{SL}_2\,\mathbb{Z})^n$ in the product space $G = (\mathrm{SL}_2\,\mathbb{R})^n$ with the one-paramter subgroup (g_t) inducing the same diagonal flow in each factor, it was shown in [4] that the set D has Hausdorff dimension $\dim G - \frac{1}{2}$, $n \geq 2$. Corollary 1.2 furnishes a higher rank example with Γ an irreducible lattice, suggesting the following

Conjecture. *For any non-quasi-unipotent flow $(G/\Gamma, g_t)$ on a finite-volume, noncompact homogeneous space, the set $D \subset G/\Gamma$ of points that lie on divergent trajectories has Hausdorff dimension strictly less than* $\dim G$. [1]

Similar results are known for Teichmüller flows that are consistent with the behavior of non-quasi-unipotent flows; for example, in [16] Masur showed that for any Teichmüller disk τ, the set of points that lie on divergent trajectories is a subset $D_\tau \subset \tau$ of Hausdorff codimension at least $\frac{1}{2}$. For some τ, the Hausdorff codimension $\frac{1}{2}$ is realised [3]. It is also known that for a generic τ, the Hausdorff codimension of D_τ is strictly less than one [18]. Likewise, the set $B_\tau \subset \tau$ of points that lie on bounded trajectories of the Teichmüller flow is shown in [12] to have $\mathrm{H.dim}\, B_\tau = \dim \tau$.

Further applications. Let $\mathrm{DI}_\delta(d)$ be the set of all $\mathbf{x} \in \mathbb{R}^d$ for which there exists T_0 such that for all $T > T_0$ the system of inequalities (1) admits an integer solution. In the course of proving Theorem 1.1 we obtain

Theorem 1.3. *There are positive constants c_1, c_2 such that*

$$\frac{4}{3} + \exp(-c_1\delta^{-4}) \leq \mathrm{H.dim}\,\mathrm{DI}_\delta(2) \leq \frac{4}{3} + c_2\delta.$$

[1] It is a well-known result due to G.A. Margulis that every quasi-unipotent flow on a finite-volume homogeneous space does not admit any divergent trajectories.

The upper bounds in Theorems 1.1 and 1.3 are obtained in §4 while the lower bounds are obtained in §5.

Corollary 1.4. *There is a compact subset $K \subset \mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ such that the set D_K of all points that lie on trajectories of (g_t) that eventually stay outside of K is a set of Hausdorff dimension strictly less than* 8.

Our techniques also allow us to answer a question of Starkov in [21] concerning the existence of divergent trajectories of the flow g_t. §6 is devoted to a proof of the following.

Theorem 1.5. *Given any function $\varepsilon(t) \to 0$ as $t \to \infty$ there is a dense set of $\mathbf{x} \in \mathrm{Sing}^*(2)$ such that $\ell(g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}) \geq \varepsilon(t)$ as $t \to \infty$.*

As a by-product of our analysis, we also obtain applications to number theory.

Theorem 1.6. *Let $\frac{\mathbf{p}_j}{q_j}, j = 0, 1, \ldots$ be the sequence of best approximations to $\mathbf{x}$ relative to some given norm $\|\cdot\|$ on $\mathbb{R}^d$. Let $\mathbf{m}_j \in \mathbb{Z}^d$ be given by $m_{j,i} = p_{j,i}q_{j+1} - p_{j+1,i}q_j$. Then*

$$\frac{\|\mathbf{m}_j\|}{q_j(q_{j+1}+q_j)} \leq \left\|\mathbf{x} - \frac{\mathbf{p}_j}{q_j}\right\| \leq \frac{2\|\mathbf{m}_j\|}{q_j q_{j+1}}.$$

This result, proved in §2, essentially generalises the fundamental inequalities satisfied by convergents of continued fractions.

The basic idea behind the proof of Theorem 1.1 is to cover $\mathrm{Sing}(d)$ by sets of the form

$$\Delta(v) = \{\mathbf{x} : v \text{ is a best approximation to } \mathbf{x}\}.$$

The diameters of these sets are approximately (see (10))

$$\frac{\delta_v}{|v|^{1+1/d}}$$

where δ_v measures the distortion of some d-dimensional lattice $\mathcal{L}(v)$ associated to v. It turns out that these lattices become very distorted for best approximations to points in $\mathrm{Sing}(d)$ (Theorem 2.17) and this gives additional control over the size of $\Delta(v)$.

To get Hausdorff dimension estimates, we use a technical device introduced in [4] (self-similar coverings) which allows one to systematically make refinements to the initial covering. Each refinement corresponds to a method of choosing subsequences of best approximations that lead to more efficient covers. As is well-known, there can be arbitrarily long (consecutive) sequences of "colinear" best approximations, i.e. lying on a common (rational) line in $\mathbb{R}^d$. (See [14].) The first

"method", or *acceleration* as we prefer to call it, is to take the subsequence of best approximations consisting of those that do not lie on the rational line determined by the previous two. This almost gives the upper bound (Proposition 4.12) except, for technical reasons, it is necessary to pass to a further subsequence where the distortion parameter δ_v is monotone. For the lower bound we construct Cantor-like subsets of $\mathrm{Sing}(d)$ and use a lower bound estimate (Theorem 3.2) involving a "spacing condition" (iii) in terms of a "local Hausdorff dimension formular" (iv). As an illustration of the basic technique, we show that the set of real numbers with divergent partial quotients is a set of Hausdorff dimension $\frac{1}{2}$ (Theorem 3.5), although accelerations are not needed for this calculation.

Perhaps the most significant contribution this paper makes to the theory of simultaneous Diophantine approximation is the introduction of the sets $\mathcal{L}(v)$, which are computable objects containing useful information about the approximation properties of the rational point. It may be helpful to think of it as an object that encodes the relative positions of other rationals in the vicinity of v. One of the main obstructions when addressing the case $d \geq 3$ is the fact that the moduli space for the lattice $\mathcal{L}(v)$ is no longer a space of rank one.

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2. Sequence of Best Approximations

Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$ and let $\|\cdot\|'$ denote the norm on $\mathbb{R}^{d+1}$ given by $\|(\mathbf{x}, y)\|' := \max(\|\mathbf{x}\|, |y|)$. For any $\mathbf{x} \in \mathbb{R}^d$, let $W_{\mathbf{x}} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$W_{\mathbf{x}}(t) = \log \ell(g_t h_{\mathbf{x}} \mathbb{Z}^{d+1})$$

where $\ell(\cdot)$ denotes the $\|\cdot\|'$-length of the shortest nonzero vector.

Lemma 2.1. *The function $W_{\mathbf{x}}$ is continuous, piecewise linear with slopes $-d$ and $+1$; moreover, it has infinitely many local minima if and only if $\mathbf{x} \notin \mathbb{Q}^d$.*

Proof. For any $v \neq 0$ the function $t \to \log \|g_t v\|'$ is a continuous, piecewise linear function with at most one critical point. Its derivative

is defined everywhere except at the critical point and is either equal to $-d$ or $+1$. For each $\tau \in \mathbb{R}$, there is a finite set $F_\tau \subset h_{\mathbf{x}}\mathbb{Z}^{d+1}$ such that $W_{\mathbf{x}}(t) = \log \ell(g_t F_\tau)$ for all t in some neighborhood of τ. Thus, $W_{\mathbf{x}}$ is continuous, piecewise linear with slopes $-d$ and $+1$, because it satisfies the same property locally.

Let C be the set of critical points of $W_{\mathbf{x}}$ and note that F_τ can be chosen so that it is constant on each connected component of $\mathbb{R} \setminus C$. If $\mathbf{x} \notin \mathbb{Q}^d$ then $\ell(g_t F_\tau) \to \infty$ as $t \to \infty$ whereas Minkowski's theorem implies $\ell(g_t h_{\mathbf{x}}\mathbb{Z}^{d+1})$ is bounded above for all t. Hence, $F_t \neq F_\tau$ for some $t > \tau$ so that $C \cap [\tau, t] \neq \varnothing$ and since τ can be chosen arbitrarily large, it follows that $W_{\mathbf{x}}$ has infinitely many local minima. If $\mathbf{x} \in \mathbb{Q}^d$ then $\mathbf{x} = \frac{\mathbf{p}}{q}$ for some $\mathbf{p} \in \mathbb{Z}^d, q \in \mathbb{Z}$ such that $v = (\mathbf{p}, q)$ satisfies $\gcd(v) = 1$. Observe that we can take $F_\tau = \{h_{\mathbf{x}} v\}$ for all sufficiently large τ. Thus, C is bounded, hence finite, and in particular $W_{\mathbf{x}}$ has at most finitely many local minima. □

Definition 2.2. Let v be a vector in

$$Q := \{(\mathbf{p}, q) \in \mathbb{Z}^{d+1} : \gcd(\mathbf{p}, q) = 1, q > 0\}$$

and τ a local minimum time of the function $W_{\mathbf{x}}$. We shall say "v realises the local minimum of $W_{\mathbf{x}}$ at time τ" if $W_{\mathbf{x}}(t) = \log \|g_t h_{\mathbf{x}} v\|'$ for all t in some neighborhood of τ. The set of vectors in Q that realise *some* local minimum of $W_{\mathbf{x}}$ will be denoted by

$$\Sigma(\mathbf{x}).$$

By convention, $\tau = +\infty$ is considered a local minimum time of $W_{\mathbf{x}}$.

Notation. Given $\mathbf{x} \in \mathbb{R}^d$ and $v = (\mathbf{p}, q) \in Q$ we let

$$\operatorname{hor}_{\mathbf{x}}(v) := \|q\mathbf{x} - \mathbf{p}\| \qquad \text{and} \qquad |v| := q. \tag{2}$$

Lemma 2.3. *Let $v \in Q$. Then $v \in \Sigma(\mathbf{x})$ if and only if for any $u \in Q$*

(i) $|u| < |v|$ *implies* $\operatorname{hor}_{\mathbf{x}}(u) > \operatorname{hor}_{\mathbf{x}}(v)$, *and*
(ii) $|u| = |v|$ *implies* $\operatorname{hor}_{\mathbf{x}}(u) \geq \operatorname{hor}_{\mathbf{x}}(v)$.

Proof. Suppose $v \in \Sigma(\mathbf{x})$, so that it realises a local minimum of $W_{\mathbf{x}}$. If $u \in Q$ does not satisfy (i) then $\|g_t h_{\mathbf{x}} u\|' \leq \|g_t h_{\mathbf{x}} v\|'$ for all t, with strict inequality for at least some t, which implies that v cannot realise a local minimum of $W_{\mathbf{x}}$, a contradiction. Therefore, (i) holds for any $u \in Q$. The argument that (ii) holds as well is similar. Conversely, suppose (i) and (ii) hold for all $u \in Q$. Let (τ, ε) be the unique local minimum of $t \to \|g_t h_{\mathbf{x}} v\|'$. Let B' be the closed $\|\cdot\|'$-ball of radius ε at the origin. Write it as $B \times I$ where $B \subset \mathbb{R}^d$ and $I \subset \mathbb{R}$. Let $Z = B' \cap g_t h_{\mathbf{x}}\mathbb{Z}^{d+1}$ and $Z^* = Z \setminus \{0\}$. Then (i) and (ii) imply $Z^* \subset \partial B \times \partial I$, which implies v realises the shortest nonzero vector at time τ. (By this we mean

$g_\tau h_{\mathbf{x}} v$ is the shortest nonzero vector in $g_\tau h_{\mathbf{x}} \mathbb{Z}^{d+1}$.) Since there is exists a slightly larger ball B'' containing B' such that $B'' \cap g_t h_{\mathbf{x}} \mathbb{Z}^{d+1} = Z$, it follows that v realises the shortest nonzero vector for an interval of t about τ. Thus, v realises a local minimum of $W_{\mathbf{x}}$ and $v \in \Sigma(\mathbf{x})$. $\square$

The next lemma was proved in [14] for the case $d = 2$.

Lemma 2.4. *If $u, v \in \Sigma(\mathbf{x})$ realise a consecutive pair of local minima of the function $W_{\mathbf{x}}$, then they span a* primitive *two-dimensional sublattice of $\mathbb{Z}^{d+1}$, i.e. $\mathbb{Z}^{d+1} \cap (\mathbb{R}u + \mathbb{R}v) = \mathbb{Z}u + \mathbb{Z}v$.*

Proof. Let $F = \{\pm u, \pm v\}$ and denote its convex hull by $\mathrm{conv}(F)$. Let $\mathcal{C}(\mathbf{x})$ be the collection of subsets of $\mathbb{R}^{d+1}$ of the form

$$C(r, h) = \{(\mathbf{a}, b) : \|\mathbf{a}\| \leq r, |b| \leq h\} \qquad r > 0,\ h > 0$$

that intersect $h_{\mathbf{x}}(\mathbb{Z}^{d+1} \setminus \{0\})$ on the boundary but not in the interior. Observe that the set of maximal (resp. minimal) elements of $\mathcal{C}(\mathbf{x})$, partially ordered by inclusion, is in one-to-one correspondence with the set of local maxima (resp. local minima) of the function $W_{\mathbf{x}}$. Since u and v realise distinct local minima, $|u| \neq |v|$. Hence, without loss of generality, assume that $|u| < |v|$. The element of $\mathcal{C}(\mathbf{x})$ corresponding to the unique local maximum of $W_{\mathbf{x}}$ between the consecutive pair of local minima determined by u and v is given by the parameters $r = \mathrm{hor}_{\mathbf{x}}(u)$ and $h = |v|$. Note that $\mathrm{conv}(h_{\mathbf{x}} F)$ is a subset of $C(r, h)$ and intersects the boundary of $C(r, h)$ in the four points of $h_{\mathbf{x}} F$. Therefore, $\mathrm{conv}(F) \cap \mathbb{Z}^{d+1} = F \cup \{0\}$, which is equivalent to u, v spanning a primitive two-dimensional sublattice of $\mathbb{Z}^{d+1}$. $\square$

Recall that $\frac{\mathbf{p}}{q} \in \mathbb{Q}^d$ is a *best approximation* to $\mathbf{x}$ if

(i) $\|q\mathbf{x} - \mathbf{p}\| < \|n\mathbf{x} - \mathbf{m}\|$ for any $(\mathbf{m}, n) \in \mathbb{Z}^{d+1}$, $0 < n < q$,
(ii) $\|q\mathbf{x} - \mathbf{p}\| \leq \|q\mathbf{x} - \mathbf{p}'\|$ for any $\mathbf{p}' \in \mathbb{Z}^d$.

The study of best approximations is central to the theory of simultaneous Diophantine approximation and has a long history going back to Lagrange, who showed that the sequence of best approximations in the case $d = 1$ are enumerated by the convergents in the continued fraction expansion. The literature on this subject is extensive. We refer the reader to the papers [13] and [15], which contain many further references.

Lemma 2.3 gives a simple dynamical interpretation for the sequence of best approximations, ordered by increasing height: *they correspond precisely to the sequence of vectors that realise the local minima of $W_{\mathbf{x}}$.*

Notation. For any $v \in Q$ let

$$\dot{v} := \frac{\mathbf{p}}{q} \in \mathbb{Q} \quad \text{where} \quad v = (\mathbf{p}, q) \in Q. \tag{3}$$

We shall often ignore the distinction between a vector $v \in Q$ and the rational $\dot{v}$ corresponding to it. Thus, we may refer to a sequence (v_j) in Q as the sequence of best approximations to $\mathbf{x}$, by which we mean for every j the vector v_j realises the jth local minimum of $W_{\mathbf{x}}$. This raises the issue of the uniqueness of the vector realising a local minimum of $W_{\mathbf{x}}$, or equivalently, the existence of best approximations to $\mathbf{x}$ of the same height. In the case $d = 1$, this can only happen if $\mathbf{x}$ is a half integer. In general, there are at most finitely many local minima that can be realised by multiple vectors in Q. (See the remark following Theorem 2.10 below.) When referring to "the sequence of best approximations to $\mathbf{x}$" we really mean *any* sequence (v_j) such that the jth local minimum of $W_{\mathbf{x}}$ is realised by v_j.

2.1. **Two-dimensional sublattices.**

Definition 2.5. For any $v \in Q$, we denote by

$$\mathcal{L}(v)$$

the set of primitive two-dimensional sublattices of $\mathbb{Z}^{d+1}$ containing v.

There is a natural way to view $\mathcal{L}(v)$ as the set of primitive elements in some d-dimensional lattice. Indeed, consider the exact sequence of real vector spaces

$$\mathbb{R} \longrightarrow \wedge^1\mathbb{R}^{d+1} \xrightarrow{\varphi} \wedge^2\mathbb{R}^{d+1} \longrightarrow \wedge^3\mathbb{R}^{d+1} \tag{4}$$

where each map is given by exterior multiplication by v. Since $v \neq 0$, the kernel of φ is one-dimensional, from which it follows that the image of φ, denoted

$$\mathcal{L}_{\mathbb{R}}(v),$$

is a real vector space of dimension d. Similarly, we have an exact sequence of free $\mathbb{Z}$-modules

$$\mathbb{Z} \longrightarrow \wedge^1\mathbb{Z}^{d+1} \longrightarrow \wedge^2\mathbb{Z}^{d+1} \longrightarrow \wedge^3\mathbb{Z}^{d+1}$$

where the image of second map is a free $\mathbb{Z}$-module of rank d, denoted

$$\mathcal{L}_{\mathbb{Z}}(v).$$

It embeds $\mathcal{L}_{\mathbb{Z}}(v)$ as a d-dimensional lattice in $\mathcal{L}_{\mathbb{R}}(v)$. The set of *oriented*, primitive, two-dimensional sublattices of $\mathbb{Z}^{d+1}$ that contain v is given by

$$\mathcal{L}_+(v) = \{u \wedge v : u, v \in Q, \mathbb{Z}u + \mathbb{Z}v \in \mathcal{L}(v)\}.$$

The next lemma shows that $\mathcal{L}_+(v)$ coincides with the set of primitive elements of lattice $\mathcal{L}_{\mathbb{Z}}(v)$.

Lemma 2.6. *Let $L = \mathbb{Z}u + \mathbb{Z}v$ be a two dimensional sublattice of $\mathbb{Z}^{d+1}$. Then L is primitive if and only if $u \wedge v$ is primitive as an element of $\mathcal{L}_{\mathbb{Z}}(v)$, i.e. $u \wedge v \neq dw$ for any $d \geq 2$ and $w \in \mathcal{L}_{\mathbb{Z}}(v)$.*

Proof. Let $L' = \mathbb{Z}^{d+1} \cap \operatorname{span} L$ so that L is primitive iff $L' = L$. Suppose $u \wedge v = dw$ for some $d \geq 2$ and $w \in \mathcal{L}_{\mathbb{Z}}(v)$. Then $w = u' \wedge v$ for some $u' \in \mathbb{Z}^{d+1}$. Since $(du' - u) \wedge v = 0$, we have $du' = u + cv$ for some $c \in \mathbb{Z}$. Since $d \geq 2$, $u' \notin L$. Hence, $L' \neq L$. Conversely, suppose $L' \neq L$. Choose $u' \in Q$ so that $L' = \mathbb{Z}u' + \mathbb{Z}v$. Since $L \subset L'$, we may write $u = au' + bv$ for some $a, b \in \mathbb{Z}$. Then $u \wedge v = au' \wedge v$. Since the index of L in L' is given by $|a|$, we have $u \wedge v = dw$ where $d = |a|$ and $w = \pm u' \wedge v \in \mathcal{L}_{\mathbb{Z}}(v)$. Since $L \neq L'$, $d \geq 2$. □

Identifying $\wedge^1 \mathbb{R}^{d+1}$ with $\mathbb{R}^{d+1}$, we note that the kernel of φ is given by the one-dimensional subspace $\mathbb{R}v$. Thus, φ induces an isomorphism of $\mathcal{L}_{\mathbb{R}}(v)$ with the space of cosets of $\mathbb{R}v$ in $\mathbb{R}^{d+1}$. The elements in $\mathcal{L}_{\mathbb{Z}}(v)$ correspond to cosets that have nonempty intersection with $\mathbb{Z}^{d+1}$. Let E_+ be the expanding eigenspace for the action of g_1. Then

$$\mathbb{R}^{d+1} = E_+ \oplus \mathbb{R}v$$

and the map that sends a coset of $\mathbb{R}v$ to the point of intersection with E_+ induces an isomorphism of of $\mathcal{L}_{\mathbb{R}}(v)$ with E_+. The norm $\|\cdot\|$ on $\mathbb{R}^d$, which is naturally identified with E_+, induces a norm on $\mathcal{L}_{\mathbb{R}}(v)$, which we shall denote by

$$\|\cdot\|_{\mathcal{L}(v)}.$$

Let E_- be the contracting eigenspace for g_1. Since $\dim E_- = 1$, the kth exterior power decomposes into two eigenspaces for g_t

$$\wedge^k \mathbb{R}^{d+1} = E_+^k \oplus E_-^k$$

where E_+^k and E_-^k are naturally identified with $\wedge^k E_+$ and $\wedge^{k-1} E_+$, respectively. Let $e_1, \ldots, e_{d+1}$ be the standard basis vectors for $\mathbb{R}^{d+1}$. The operation of wedging with e_{d+1} induces an isomorphism between E_+^k and E_-^{k+1}; in particular, we have an isomorphism of E_-^2 with E_+^1, which is naturally identified with $\mathbb{R}^d$ through the isomorphisms with E_+. The norm $\|\cdot\|$ on $\mathbb{R}^d$ induces a norm on E_-^2, which may be extended to a seminorm on all of $\wedge^2 \mathbb{R}^{d+1}$ by defining the (semi)norm of an element to be the norm of the component in E_-^2. This seminorm will be denoted by

$$|\cdot|.$$

Given $L \in \mathcal{L}(v)$ we may form an element $u \wedge v \in \mathcal{L}_{\mathbb{Z}}(v)$ by choosing any pair in Q such that $L = \mathbb{Z}u + \mathbb{Z}v$. This element is well-defined up to sign, so it makes sense to talk about the norm of L an element in $\mathcal{L}_{\mathbb{R}}(v)$ and also as an element of $\wedge^2\mathbb{R}^{d+1}$; denote these, respectively, by

$$\|L\|_{\mathcal{L}(v)} \quad \text{and} \quad |L|.$$

There is a simple relation between these norms. Note that the action of $h_{\mathbf{x}}$ on an element in $\wedge^2\mathbb{R}^{d+1}$ preserves the component in E_-^2. Now let u', v' be the respective images of u, v under $h_{\dot{v}}$. Then $u' \wedge v' = |v| u'_+ \wedge e_{d+1}$ where u'_+ is the component of u' in E_+. Note that u'_+ is precisely the point where the coset of $\mathbb{R}v$ corresponding to L intersects E_+. Note also that its norm is given by $\mathrm{hor}_{\dot{v}}(u)$. It follows that

$$\|L\|_{\mathcal{L}(v)} = \mathrm{hor}_{\dot{v}}(u) = \frac{|L|}{|v|}. \tag{5}$$

The image of $\mathcal{L}_{\mathbb{Z}}(v)$ under the isomorphism of $\mathcal{L}_{\mathbb{R}}(v)$ with E_+ is simply the image of $\mathbb{Z}^{d+1}$ under the projection of $\mathbb{R}^{d+1} = E_+ \oplus \mathbb{R}v$ onto E_+, i.e. the projection along lines parallel to v. Alternatively, it can be described as the set of all components in E_+ of vectors in $h_{\dot{v}}\mathbb{Z}^{d+1}$. Its volume is given by

$$\mathrm{vol}\left(\mathcal{L}_{\mathbb{Z}}(v)\right) = \frac{1}{|v|}.$$

Since $\mathcal{L}_+(v)$ is a discrete subset of a normed vector space, there exists an element of minimal positive norm. While this element may not be unique, we shall choose one for each $v \in Q$, once and for all, and denote it by

$$L(v).$$

The corresponding element in $\mathcal{L}(v)$ will be denoted by the same symbol.[2] Since the norm of the shortest nonzero vector in a unimodular lattice in $\mathbb{R}^d$ is bounded above by some constant μ_0 (depending on $\|\cdot\|$) we have for all $v \in Q$

$$\frac{|L(v)|}{|v|^{1-1/d}} \le \mu_0. \tag{6}$$

Let us mention a form of (5) that is symmetric with respect to u and v. Let $\mathrm{dist}(\cdot,\cdot)$ denote the metric on $\mathbb{R}^d$ induced by the norm $\|\cdot\|$. Then $\mathrm{hor}_{\dot{v}}(u) = |u| \,\mathrm{dist}(\dot{u},\dot{v})$ so that

$$\mathrm{dist}(\dot{u},\dot{v}) = \frac{|u \wedge v|}{|u||v|}. \tag{7}$$

[2]We shall often blur the distinction between elements in $\mathcal{L}_+(v)$ and $\mathcal{L}(v)$, leaving it to the context to determine which meaning is intended.

Let us extend the notation $\dot{v}, |v|$ and $\mathrm{hor}_{\mathbf{x}}(v)$ introduced in (3) and (2) to the set $E_+^c = \mathbb{R}^{d+1} \setminus E_+$. Then (7) holds for all $u, v \in E_+^c$. It will be convenient to allow overscripts on the arguments of $\mathrm{dist}(\cdot,\cdot)$ to be dropped; formally, we are extending $\mathrm{dist} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to a function that is defined on $E \times E$ where E is the disjoint union of $\mathbb{R}^d$ and E_+^c. These conventions allow for more appealing formulas such as

$$|u \wedge v| = |u||v| \,\mathrm{dist}(u, v), \quad \mathrm{hor}_{\mathbf{x}}(v) = |v| \,\mathrm{dist}(v, \mathbf{x}).$$

2.2. **Domains of approximation.** We now investigate the sets

$$\Delta(v) := \{\mathbf{x} \in \mathbb{R}^d : v \in \Sigma(\mathbf{x})\}$$

for $v \in Q$. By Lemma 2.3,

$$\Delta(v) = \big(\cap_{|u|<|v|} \Delta_u(v)\big) \cap \big(\cap_{|u|=|v|} \overline{\Delta_u(v)}\big)$$

where the sets $\Delta_u(v)$, defined only for $u \in Q \setminus \{v\}$, are given by

$$\Delta_u(v) = \{\mathbf{x} \in \mathbb{R}^d : \mathrm{hor}_{\mathbf{x}}(u) > \mathrm{hor}_{\mathbf{x}}(v)\}.$$

We note that $\Delta_u(v)$ is bounded iff $|u| < |v|$.

Lemma 2.7. *For any (distinct) $u, v \in Q$ with $|u| \leq |v|$ we have*

$$\mathrm{dist}(\mathbf{x}, u) > \mathrm{dist}(u + v, u) \quad \forall \mathbf{x} \in \Delta_u(v).$$

Here, $\mathrm{dist}(u + v, \cdot)$ *means* $\mathrm{dist}(\dot{w}, \cdot)$ *where $w = u + v$.*

Proof. By definition, $\mathbf{x} \in \Delta_u(v)$ iff

$$\mathrm{dist}(v, \mathbf{x}) < \lambda \,\mathrm{dist}(u, \mathbf{x}) \quad \text{where} \quad \lambda = \frac{|u|}{|v|} \leq 1.$$

Let $w = u + v$. Since $\mathrm{dist}(u, v) = \mathrm{dist}(u, w) + \mathrm{dist}(w, v)$ and

$$\mathrm{dist}(v, w) = \frac{|v \wedge w|}{|v||w|} = \frac{|u|}{|v|} \left(\frac{|u \wedge w|}{|u||w|} \right) = \lambda \,\mathrm{dist}(u, w)$$

the triangle inequality implies

$$(1 + \lambda) \,\mathrm{dist}(u, w) \leq \mathrm{dist}(u, \mathbf{x}) + \mathrm{dist}(v, \mathbf{x}) < (1 + \lambda) \,\mathrm{dist}(u, \mathbf{x})$$

and the lemma follows. □

Remark 2.8. It follows easily from Lemma 2.7 that the infimum of

$$\{\mathrm{dist}(u, \mathbf{x}) : \mathbf{x} \in \overline{\Delta_u(v)}\}$$

is realised at the rational point corresponding to $u+v$. It can similarly be shown that the supremum is realised by the rational corresponding to $v - u$. In the case when $\|\cdot\|$ is the Euclidean norm, the set $\Delta_u(v)$ is the open Euclidean ball having these points as antipodal points.

Lemma 2.9. *Given $L \in \mathcal{L}_+(v)$ there are unique vectors $u_\pm \in Q$ satisfying $L = u_\pm \wedge v$ and $|u_\pm| \le |v|$. Moreover, $|u_+| < |v|$ iff $v = u_+ + u_-$ iff $|u_-| < |v|$. Similarly, $|u_+| = |v|$ iff $2v = u_+ + u_-$ iff $|u_-| = |v|$.*

Proof. Existence and unique of $u_\pm$ is clear. If $|u_+| < |v|$ then $v - u_+$ satisfies the conditions defining u_- so that $v - u_+ = u_-$. Similarly, if $|u_+| = |v|$ then $2v - u_+ = u_-$. □

Let $B(\mathbf{x}, r) \subset \mathbb{R}^d$ denote the open ball at $\mathbf{x}$ of radius r.

Theorem 2.10. *Given $v \in Q$ with $|v| > 1$, let $r = \frac{|L(v)|}{|v|^2}$. Then*

$$B(\dot{v}, \frac{r}{2}) \subset \Delta(v) \subset B(\dot{v}, 2r). \tag{8}$$

Proof. Let $L \in \mathcal{L}(v)$ and choose the orientation for it such that $u_\pm$ in Lemma 2.9 satisfy $|u_+| \ge |u_-|$. For any $\mathbf{x} \in \Delta_{u_-}(v)$ we have

$$\operatorname{dist}(\mathbf{x}, v) < \frac{|u_-|}{|v|} \operatorname{dist}(\mathbf{x}, u_-) < \frac{|u_-|}{|v|} \left(\operatorname{dist}(\mathbf{x}, v) + \frac{|L|}{|u_-||v|} \right)$$

so that

$$\left(1 - \frac{|u_-|}{|v|}\right) \operatorname{dist}(\mathbf{x}, v) < \frac{|L|}{|v|^2}.$$

If $|u_+| < |v|$ then it follows that

$$\operatorname{dist}(\mathbf{x}, v) < \frac{|L|}{|u_+||v|} \le \frac{2|L|}{|v|^2}. \tag{9}$$

We claim L could have been chosen initially so that $|u_+| < |v|$. Indeed, let $B'(r') = \overline{B(0, r')} \times (-|v|, |v|)$ and note that since $|v| > 1$ there is a smallest $r' > 0$ such that $B'(r') \cap h_{\dot{v}} \mathbb{Z}^{d+1} \not\subset E_+$. Then $Q \cap h_{-\dot{v}} B'(r') \neq \varnothing$ and the desired property for L is satisfied by $\mathbb{Z}u + \mathbb{Z}v$ for any u in this set. This proves the claim, and hence $\Delta(v) \subset B(\dot{v}, 2r)$.

For the other inclusion, consider $u \in Q$ with $|u| \le |v|$ and $u \neq v$. Let $\lambda = \frac{|u|}{|v|}$ and $\mu > 0$. For any $\mathbf{x} \in B(\dot{v}, \mu r)$ we have

$$\operatorname{dist}(\mathbf{x}, v) < \frac{\mu |L|}{|v|^2} \le \lambda\mu \frac{|u \wedge v|}{|u||v|}$$

whereas

$$\operatorname{dist}(\mathbf{x}, u) > (1 - \lambda\mu) \frac{|u \wedge v|}{|u||v|}.$$

Thus, $\mathbf{x} \in \Delta_u(v)$ provided

$$\mu \le 1 - \lambda\mu$$

and since $\lambda \le 1$, this holds for $\mu = \frac{1}{2}$. □

As a corollary of Theorem 2.10 we get

$$\operatorname{diam} \Delta(v) \asymp \frac{|L(v)|}{|v|^2} \leq \frac{\mu_0}{|v|^{1+1/d}}. \tag{10}$$

where μ_0 is the constant satisfying (6). Here, $A \asymp B$ means $C^{-1}B \leq A \leq CB$ for some constant C.

Remark 2.11. Observe that if $u, v \in Q$ are distinct vectors that realise the same local minimum of $W_{\mathbf{x}}$, then $|u| = |v|$ so that we can think of $u - v$ as an element of $\mathbb{Z}^d$; moreover,

$$\|u - v\| = |v| \operatorname{dist}(u, v) \leq \frac{8\mu_0}{|v|^{1/d}}.$$

If we let λ_0 be the norm of the shortest nonzero vector in $\mathbb{Z}^d$, then it follows that any vector $v \in \Sigma(\mathbf{x})$ satisfying

$$|v| > \frac{8^d \mu_0^d}{\lambda_0^d}$$

is uniquely determined by the local minimum that it realises. In other words, the sequence of best approximations is eventually uniquely determined. This fact had already been observed in [13].

Theorem 2.12. *Let $v \in \Sigma(\mathbf{x})$ and suppose $u \in Q$ is such that $|u| < |v|$. Then*

$$\frac{1}{2} \operatorname{dist}(u, v) < \operatorname{dist}(u, \mathbf{x}) < 2 \operatorname{dist}(u, v). \tag{11}$$

Proof. Let $L \in \mathcal{L}(v)$ be the one containing u. Then $|u \wedge v| = b|L|$ for some positive integer b. By Lemma 2.7 we have

$$\operatorname{dist}(\mathbf{x}, u) > \operatorname{dist}(u + v, u) = \frac{b|L|}{|u + v||u|} \geq \frac{b|L|}{2|u||v|} = \frac{1}{2} \operatorname{dist}(u, v).$$

From the *first* inequality in (9) we have

$$\begin{aligned} \operatorname{dist}(\mathbf{x}, u) &\leq \operatorname{dist}(u, v) + \operatorname{dist}(v, \mathbf{x}) \\ &< \frac{b|L|}{|u||v|} + \frac{|L|}{|u_+||v|} = \left(1 + \frac{|u|}{b|u_+|}\right) \operatorname{dist}(u, v). \end{aligned}$$

If $b \geq 2$ the expression in parentheses is at most 2. If $b = 1$ then $u = u_\pm$ and the same is true again. Thus, $\operatorname{dist}(\mathbf{x}, u) < 2 \operatorname{dist}(u, v)$. $\square$

Theorem 1.6 is a consequence of the following.

Theorem 2.13. *Let $\frac{\mathbf{p}_j}{q_j}$ be the sequence of best approximations to $\mathbf{x}$ and set $v_j = (\mathbf{p}_j, q_j)$ and $L_{j+1} = \mathbb{Z}v_{j+1} + \mathbb{Z}v_j$. Then*

$$\frac{|L_{j+1}|}{q_j(q_{j+1}+q_j)} \leq \left\| \mathbf{x} - \frac{\mathbf{p}_j}{q_j} \right\| \leq \frac{2|L_{j+1}|}{q_j q_{j+1}}. \tag{12}$$

Proof. The first (resp. second) inequality follows by applying Lemma 2.7 (resp. Theorem 2.12) with $u = v_j$ and $v = v_{j+1}$. $\square$

It can be shown that $|L_j| \leq C|L(v_j)|$ for all j (and $\mathbf{x}$) where C is a constant that depends only on the norm $\|\cdot\|$. Using this, we may rewrite (12) as

$$\|q_j \mathbf{x} - \mathbf{p}_j\| \asymp \frac{\delta(v_{j+1})}{q_{j+1}^{1/d}}$$

where

$$\delta(v) = |v|^{1/d} \|L(v)\|_{\mathcal{L}(v)} \leq \mu_0. \tag{13}$$

As we shall see $\mathbf{x} \in \mathrm{Sing}(d)$ if and only if $\delta(v_j) \to 0$ as $j \to \infty$. See Theorem 2.17 below.

2.3. **Characterisation of singular vectors.** The sequence of critical times of the function $W_{\mathbf{x}}$ are ordered by

$$\tau_0 < t_0 < \tau_1 < t_1 < \dots$$

where τ_j (resp. t_j) is the jth local maximum (resp. minimum) time. Note that the first critical point τ_0 is a local maximum because as $t \to -\infty$ we have $W_{\mathbf{x}}(t) = t + \log \|v_{-1}\|$ where v_{-1} is any nonzero vector in $\mathbb{Z}^d$ of minimal $\|\cdot\|$-norm.

Definition 2.14. For $u, v \in Q$ and $\mathbf{x} \in \mathbb{R}^d$, let $\varepsilon_{\mathbf{x}}(u, v)$ be defined by

$$\varepsilon_{\mathbf{x}}(u, v)^{1+1/d} = |v|^{1/d} \operatorname{hor}_{\mathbf{x}}(u).$$

Lemma 2.15. *For any $u, v \in Q$ with $|u| < |v|$ and for any $\mathbf{x} \in \Delta(v)$ there is a unique time τ when $\|g_\tau h_{\mathbf{x}} u\|' = \|g_\tau h_{\mathbf{x}} v\|'$. Moreover, the common length is given by $\varepsilon_{\mathbf{x}}(u, v)$.*

Proof. Since $|u| < |v|$ and $\mathbf{x} \in \Delta(v)$ we have $\operatorname{hor}_{\mathbf{x}}(u) > \operatorname{hor}_{\mathbf{x}}(v)$. This implies the existence of τ. Let $\varepsilon = \varepsilon_{\mathbf{x}}(u, v)$. The point $(\tau, \log \varepsilon)$ is where the lines $y = -d(t - a)$ and $y = (t - b)$ meet, where $a = \frac{1}{d} \log |v|$ and $b = -\log \operatorname{hor}_{\mathbf{x}}(u)$. Since

$$(t, y) = \left(\frac{ad + b}{d + 1}, \frac{ad - bd}{d + 1} \right)$$

we have

$$\log\varepsilon = \frac{1}{d+1}\log|v| + \frac{d}{d+1}\log\operatorname{hor}_{\mathbf{x}}(u),$$
$$\tau = \frac{1}{d+1}\log|v| - \frac{1}{d+1}\log\operatorname{hor}_{\mathbf{x}}(u)$$

from which it follows that

$$\varepsilon^{1+1/d} = |v|^{1/d}\operatorname{hor}_{\mathbf{x}}(u) \quad\text{and}\quad e^{-(d+1)\tau} = \frac{\operatorname{hor}_{\mathbf{x}}(u)}{|v|}.$$

□

Lemma 2.16. *Assume $\mathbf{x} \notin \mathbb{Q}^d$ and set $\delta_j = \varepsilon_{\mathbf{x}}(v_{j-1}, v_j)^{1+1/d}$ where (v_j) is the sequence of best approximations to $\mathbf{x}$. Then $\mathbf{x} \in \mathrm{DI}_\delta(d)$ iff $\delta_j < \delta$ for all sufficiently large j.*

Proof. Write $v_j = (\mathbf{p}_j, q_j)$ so that Lemma 2.15 implies

$$\|q_j\mathbf{x} - \mathbf{p}_j\| = \operatorname{hor}_{\mathbf{x}}(v_j) = \frac{\delta_{j+1}}{q_{j+1}^{1/d}}.$$

If $\delta_j < \delta$ then $(\mathbf{p}_j, q_j)$ solves (1) for all $q_j < T \le q_{j+1}$. It follows that $\delta_j < \delta$ for all large enough j implies $\mathbf{x} \in \mathrm{DI}_\delta(d)$. Conversely, suppose $\mathbf{x} \in \mathrm{DI}_\delta(d)$ so that there exists T_0 such that (1) admits a solution for all $T > T_0$. Suppose j is large enough so that $q_{j+1} > T_0$. Let $(\mathbf{p}, q)$ be a solution to (1) for $T = q_{j+1}$. Then $q < q_{j+1}$, implying that

$$\|q_j\mathbf{x} - \mathbf{p}_j\| \le \|q\mathbf{x} - \mathbf{p}\| < \frac{\delta}{q_{j+1}^{1/d}}$$

from which it follows that $\delta_j < \delta$. □

The next theorem gives a characterisation is purely in terms of the sequence of best approximations, without explicit reference to $\mathbf{x}$. This will be needed to reduce the main task to a problem in symbolic dynamics.

Theorem 2.17. *Assume $\mathbf{x} \notin \mathbb{Q}^d$ and for each $j \ge 1$ set*

$$\varepsilon_j^{1+1/d} = \frac{|v_{j-1} \wedge v_j|}{|v_j|^{1-1/d}}$$

where $(v_j)_{j\ge 0}$ is the sequence of best approximations to $\mathbf{x}$. Assume $\delta > 0$ and $\varepsilon > 0$ be related by $\delta = \varepsilon^{1+1/d}$. Then $\mathbf{x} \in \mathrm{DI}_{\delta/2}(d)$ implies $\varepsilon_j < \varepsilon$ for all sufficiently large j, which in turn implies $\mathbf{x} \in \mathrm{DI}_{2\delta}(d)$. In particular, $\mathbf{x} \in \mathrm{Sing}(d)$ iff $\varepsilon_j \to 0$ as $j \to \infty$.

Proof. By Lemmas 2.16 is suffices to show

$$\frac{1}{2}\varepsilon_j^{1+1/d} \le \varepsilon_{\mathbf{x}}(v_{j-1}, v_j) \le 2\varepsilon_j^{1+1/d}$$

which holds by Theorem 2.12. □

3. Self-similar coverings

Let X be a metric space and J a countable set. Given $\sigma \subset J \times J$ and $\alpha \in J$ we let $\sigma(\alpha)$ denote the set of all $\alpha' \in J$ such that $(\alpha, \alpha') \in \sigma$. We say a sequence (α_j) of elements in J is *σ-admissible* if $\alpha_{j+1} \in \sigma(\alpha_j)$ for all j, and let J^σ denote the set of all σ-admissible sequences in J. By a *self-similar covering* of X we mean a triple $(\mathcal{B}, J, \sigma)$ where $\mathcal{B}$ is a collection of bounded subsets of X, J a countable index set for $\mathcal{B}$, and $\sigma \subset J \times J$ such that there is a map $\mathcal{E} : X \to J^\sigma$ that assigns to each $x \in X$ a σ-admissible sequence $(\alpha_j^{\mathbf{x}})$ such that for all $x \in X$

(i) $\cap B(\alpha_j^{\mathbf{x}}) = \{x\}$, and
(ii) $\operatorname{diam} B(\alpha_j^{\mathbf{x}}) \to 0$ as $j \to \infty$,

where $B(\alpha)$ denotes the element of $\mathcal{B}$ indexed by α.

Theorem 3.1. *([4], Theorem 5.3) If X is a metric space that admits a self-similar covering $(\mathcal{B}, J, \sigma)$, then* $\operatorname{H.dim} X \le s(\mathcal{B}, J, \sigma)$ *where*

$$s(\mathcal{B}, J, \sigma) = \sup_{\alpha \in J} \inf \left\{ s > 0 : \sum_{\alpha' \in \sigma(\alpha)} \left(\frac{\operatorname{diam} B(\alpha')}{\operatorname{diam} B(\alpha)} \right)^s \le 1 \right\}. \tag{14}$$

In many applications, X is a subset of some ambient metric space Y and we are given a self-similar covering $(\mathcal{B}, J, \sigma)$ of X by bounded subsets of Y rather than X. For any bounded subset $B \subset Y$ we have $\operatorname{diam}_X B \cap X \le \operatorname{diam}_Y B$ but equality need not hold in general. To compute $s(cB, J, \sigma)$ one would also need an inequality going in the other direction. While such an inequality may not be difficult to obtain, it is both awkward and unnecessary: *Theorem 3.1 remains valid in this more general situation if the diameters in (14) are taken with respect to the metric of Y.* The proof of this more general statement does not follow directly from Theorem 3.1, but the argument given in [4] applies with essentially no change and will not be repeated here.

For lower bounds, we shall use

Theorem 3.2. *Let $\mathcal{B}$ be a collection of nonempty compact subsets of a metric space Y indexed by a countable set J. Suppose $X \subset Y$ and $\sigma \subset J \times J$ are such that*

(i) *for each $\alpha \in J$, $\sigma(\alpha)$ is a finite subset of J with at least 2 elements and for each $\alpha' \in \sigma(\alpha)$ we have $B(\alpha') \subset B(\alpha)$,*

(ii) *for each* $(\alpha_j) \in J^\sigma$, *we have* $\operatorname{diam} B(\alpha_j) \to 0$ *and the unique point in* $\cap B(\alpha_j)$ *belongs to* X,

(iii) *there exists* $\rho > 0$ *such that for any* $\alpha \in J$ *and for any distinct pair* $\alpha', \alpha'' \in \sigma(\alpha)$

$$\operatorname{dist}(B(\alpha'), B(\alpha'')) > \rho \operatorname{diam} B(\alpha), \tag{15}$$

(iv) *there exists* $s > 0$ *such that for every* $\alpha \in J$,

$$\sum_{\alpha' \in \sigma(\alpha)} [\operatorname{diam} B(\alpha')]^s \geq [\operatorname{diam} B(\alpha)]^s. \tag{16}$$

Then $\text{H.dim}\, X \geq s$.

Before giving a proof of Theorem 3.2, let us discuss the significance of the spacing condition (iii). There are many theorems in the literature that can be applied to give lower bounds on $\text{H.dim}\, X$ in the setup of Theorem 3.2. Different assumptions on the set X lead to different lower bound estimates. Consider the following *distorted* Cantor set

$$C_\delta, \quad 0 < \delta < 1$$

defined as the intersection $\cap_{k \geq 0} E_k$ where $E_0 = [0,1]$, $E_1 = [0, \frac{\delta}{2}] \cup [\frac{1}{2}, 1]$ and for each $k \geq 2$, the set E_k is a disjoint union of 2^k closed intervals obtained by removing a similar $\frac{1-\delta}{2}$ fraction from each of the intervals of E_{k-1}. For each interval I of E_k the density of the intervals of E_{k+1} in I is a constant

$$\Delta_k = \frac{1+\delta}{2}$$

independent of both I and k. The length of each interval in E_k is bounded between

$$d_k^- = \frac{\delta^k}{2^k} \quad \text{and} \quad d_k^+ = \frac{1}{2^k}.$$

The length of the smallest gap between the intervals of E_k is

$$\varepsilon_k = \frac{(1-\delta) d_{k-1}^-}{2}$$

and each interval of E_{k-1} has exactly

$$m_k = 2$$

intervals of E_k contained in it. The estimate in [17] based on density gives a lower bound

$$h_d(\delta) = 1 - \limsup_{k \to \infty} \frac{\sum_{i=1}^{k+1} |\log \Delta_i|}{|\log d_k^+|} = \frac{\log(1+\delta)}{\log 2}$$

whereas the estimate in [7] (p.64) based on gaps gives

$$h_g(\delta) = \liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log m_k \varepsilon_k} = \frac{\log 2}{\log(2/\delta)}.$$

Both functions increase from 0 to 1 as δ ranges from 0 to 1, and since

$$h_d'(0+) = \frac{1}{\log 2}, \qquad h_d'(1-) = \frac{1}{2\log 2},$$
$$h_g'(0+) = \infty, \qquad h_g'(1-) = \frac{1}{\log 2},$$

we have

$$\lim_{\delta\to 0^+} \frac{h_d(\delta)}{h_g(\delta)} = 0, \quad \text{and} \quad \lim_{\delta\to 1^-} \frac{h_d(\delta)}{h_g(\delta)} = 2.$$

Note that the lower bounds are comparable in the limit as δ approaches one, but the one using a gap hypothesis is infinitely better in the limit as δ approaches zero.[3]

The exact value $h(\delta)$ for the Hausdorff dimension of C_δ is given by the unique $0 < s < 1$ satisfying

$$\left(\frac{\delta}{2}\right)^s + \left(\frac{1}{2}\right)^s = 1, \qquad \text{or} \qquad 2^s = 1 + \delta^s. \tag{17}$$

This follows from the easily verified fact that the Hausdorff measure of the set C_δ in dimension $h(\delta)$ is equal to one. Note that Theorems 3.1 and 3.2 both give $h(\delta)$ as upper and lower bounds, respectively.

From (17) we have

$$s = \log_2(1 + \delta^s) \asymp \delta^s$$

so that

$$\begin{aligned} \log s &= s \log \delta + O(1), \\ \frac{1}{s} \log \frac{1}{s} &\simeq \log \frac{1}{\delta}, \\ \frac{1}{s} &\simeq \log \frac{1}{\delta} \log \frac{1}{\delta}, \end{aligned}$$

where $A \simeq B$ means the ratio tends to one as $\delta \to 0$. It follows that

$$h(\delta) \simeq \frac{\log\log(1/\delta)}{\log(1/\delta)}. \tag{18}$$

It is easy to check that the graph of $h(\delta)$ is *nearly flat* as $\delta \to 1$ in the sense that all one-sided derivatives vanish there. Likewise, it is *nearly vertical* as $\delta \to 0$ in the sense that the graph of the inverse function is nearly flat at the origin. Thus, we see that the both estimates $h_d(\delta)$

[3] The graphs of h_d and h_g cross near the point $(.2726604^+, .3478475^+)$.

and $h_g(\delta)$ leave plenty of room for improvement in the either limiting case $\delta \to 0$ or $\delta \to 1$. One reason why $h_d(\delta)$ and $h_g(\delta)$ fail to provide sharp lower bounds in the case of C_δ may be explained by the fact that the lengths the intervals (or the gaps) in E_k are not uniformly bounded:

$$\lim_{k\to\infty} \frac{d_k^+}{d_k^-} = \infty.$$

It seems clear that better estimates should be obtainable if the parameters Δ_k, $d_k^\pm$, m_k, ε_k were replaced by parameters that also depended on the particular interval in E_k; in other words, the new parameters would be functions on a tree of intervals. Estimates on Hausdorff dimension would then be given in terms of "local conditions" to be met at each node of this tree. An example of such condition is given by (16).

Theorem 3.2 is a special case of the next theorem, which allows for a spacing condition with *weights*.

Theorem 3.3. *Suppose (i) and (ii) of Theorem 3.2 holds, and there exists a function $\rho : J \to (0,1)$ such that*

(iii') *for any $\alpha \in J$ and for any distinct pair $\alpha', \alpha'' \in \sigma(\alpha)$*

$$\operatorname{dist}(B(\alpha'), B(\alpha'')) > \rho(\alpha) \operatorname{diam} B(\alpha), \tag{19}$$

(iv') *there exists $s > 0$ such that for every $\alpha \in J$,*

$$\sum_{\alpha' \in \sigma(\alpha)} [\rho(\alpha') \operatorname{diam} B(\alpha')]^s \geq [\rho(\alpha) \operatorname{diam} B(\alpha)]^s.$$

Then H.dim $X \geq s$.

Proof. Fix any $\alpha_0 \in J$ and let

$$E = E(\alpha_0)$$

be the set of all $x \in X$ such that $\cap B(\alpha_j) = \{x\}$ for some σ-admissible sequence (α_j) starting with α_0. Let $J_0 = \{\alpha_0\}$ and $J_k = \cup_{\alpha \in J_{k-1}} \sigma(\alpha)$ for $k > 0$. Note that

$$E = \cap_{k\geq 0} E_k \quad \text{where} \quad E_k = \cup_{\alpha \in J_k} B(\alpha).$$

Since each J_k is finite, E is compact. Let $J' = \cup_{k\geq 0} J_k$.

Claim. For any finite subset $F \subset J'$ such that $\mathcal{B}_F = \{B(\alpha)\}_{\alpha\in F}$ covers E we have

$$\sum_{\alpha\in F} [\rho(\alpha) \operatorname{diam} B(\alpha)]^s \geq [\rho(\alpha_0) \operatorname{diam} B(\alpha_0)]^s. \tag{20}$$

To prove the claim, it is enough to consider the case where $\mathcal{B}_F$ has no redundant elements, i.e. $B(\alpha) \cap E \neq \varnothing$ for all $\alpha \in F$, and $B(\alpha) \not\subset B(\alpha')$

for any distinct pair $\alpha, \alpha' \in F$. It follows by (i) that the elements of $\mathcal{B}_F$ form a disjoint collection.

Proceed by induction on the smallest k such that $F \subset J_0 \cup \cdots \cup J_k$. If $k = 0$, then $F = \{\alpha_0\}$ and (20) holds with equality. For $k > 0$, first note that for any $\alpha' \in F \cap J_k$ we have $\sigma(\alpha) \subset F$ where α is the unique element of J_{k-1} such that $\alpha' \in \sigma(\alpha)$. Indeed, given $\alpha'' \in \sigma(\alpha)$, $B(\alpha'') \cap E \neq \varnothing$ implies that $B(\alpha'')$ intersects $B(\alpha''')$ for some $\alpha''' \in F$. We cannot have $B(\alpha'') \subset B(\alpha''')$ for otherwise $B(\alpha')$ would be a redundant element in $\mathcal{B}_F$; therefore, $\alpha''' \notin J_i$ for any $i < k$. Since $\alpha''' \in F$, we have $\alpha''' \in J_k$ so that $\alpha'' = \alpha''' \in F$.

Let $F' = F \cap (J_0 \cup \cdots \cup J_{k-1})$ and let $\tilde{F}$ be the subset of J_{k-1} such that $F \cap J_k$ is the disjoint union of $\sigma(\alpha)$ as α ranges over the elements of $\tilde{F}$. Then (iv) implies

$$\sum_{\alpha \in F} [\rho(\alpha) \operatorname{diam} B(\alpha)]^s \geq \sum_{\alpha \in F' \cup \tilde{F}} [\rho(\alpha) \operatorname{diam} B(\alpha)]^s$$

and the claim follows by the induction hypothesis applied to $F' \cup \tilde{F}$.

Now suppose $\mathcal{U}$ is a covering of E by open balls of radius at most ε. Since E is compact, there is a finite subcover $\mathcal{U}_0$ and without loss of generality we may assume each element of $\mathcal{U}_0$ contains some point of E. For each $U \in \mathcal{U}_0$ let (α_k) be the sequence determined by a choice of $x \in U \cap E$ and requiring $x \in B(\alpha_k), \alpha_k \in J_k$ for all k. Let k_0 be the largest index k such that $U \cap E \subset B(\alpha_k)$. Then there are distinct elements $\alpha', \alpha'' \in \sigma(\alpha_k)$ such that U intersects both $B(\alpha')$ and $B(\alpha'')$ so that (iii) implies

$$\operatorname{diam} U \geq \operatorname{dist}(B(\alpha'), B(\alpha'')) \geq \rho(\alpha_k) \operatorname{diam} B(\alpha_k).$$

Let F be the collection of α_k associated to $U \in \mathcal{U}_0$. Then

$$\sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s \geq \sum_{\alpha \in F} [\rho(\alpha) \operatorname{diam} B(\alpha)]^s \geq [\rho(\alpha_0) \operatorname{diam} B(\alpha_0)]^s.$$

Since $\varepsilon > 0$ was arbitrary, it follows that E has positive s-dimensional Hausdorff measure, and therefore, $\text{H.dim}\, X \geq \text{H.dim}\, E \geq s$. $\square$

3.1. **Divergent partial quotients.** As an illustration of the use of Theorems 3.1 and 3.2 we give an application to number theory. The rest of this section is independent of the other parts of the paper and may be skipped without loss of continuity.

We say an real number has *divergent partial quotients* if it is irrational and the sequence of terms a_k in its continued fraction expansion tends to infinity as $k \to \infty$. Let D_∞ be the set of real numbers with divergent partial quotients. Our goal is to determine the Hausdorff dimension of D_∞.

Let D_N be the set of all irrational numbers whose sequence of partial quotients satisfy $a_k > N$ for all sufficiently large k. Given $p/q \in \mathbb{Q}$ with $q \geq 2$ let $p_-/q_- < p_+/q_+$ be the convergents that precede p/q in the two possible continued fraction expansions for p/q. They are determined by the conditions

$$p_\pm q - p q_\pm = \pm 1, \quad 0 < q_\pm < q$$

and we note that $q = q_+ + q_-$. Let $v = (p,q)$ and set

$$I_N(v) = \left[\frac{Np + p_-}{Nq + q_-}, \frac{Np + p_+}{Nq + q_+} \right].$$

This interval consists of all real numbers that have p/q as a convergent and such that the next partial quotient is at least N. (In particular, we note $I_1(v) = \overline{\Delta(v)}$.) We have

$$\begin{aligned} \left|I_N(v)\right| &= \left| \frac{Np + p_-}{Nq + q_-} - \frac{p}{q} \right| + \left| \frac{p}{q} - \frac{Np + p_+}{Nq + q_+} \right| \\ &= \frac{1}{(Nq + q_-)q} + \frac{1}{(Nq + q_+)q} = \frac{2N + 1}{(Nq + q_-)(Nq + q_+)} \end{aligned}$$

so that

$$\frac{2}{(N+1)q^2} \leq \left|I_N(v)\right| \leq \frac{2}{Nq^2}.$$

Let $\mathcal{B}_N$ be the collection of intervals $I_N(v)$ for $v \in Q$, $|v| \geq 2$. Let $\sigma_N(v)$ be the set of all $v' \in Q$ of the form $av + v_\pm$ where $v_\pm = (p_\pm, q_\pm)$ and $a > N$. Then $(\mathcal{B}_N, Q, \sigma_N)$ is a self-similar covering of D_N: the map $\mathcal{E}$ is realised by sending $x \in D_N$ to a tail of the sequence of convergents of x. For any $v' \in \sigma_N(v)$ we have

$$\frac{N}{(a+1)^2(N+1)} \leq \frac{\left|I_N(v')\right|}{\left|I_N(v)\right|} \leq \frac{N+1}{a^2 N}$$

where a is the greatest integer less than $\frac{|v'|}{|v|}$. Note that there are two elements of $\sigma(v)$ associated with each $a > N$.

To estimate Hausdorff dimension we need to consider the expression

$$\sum_{v' \in \sigma(v)} \frac{\left|I_N(v')\right|^s}{\left|I_N(v)\right|^s}.$$

For any $0 < s < 1$ we have

$$\sum_{a > N} \frac{2(N+1)^s}{a^{2s} N^s} \leq \frac{4}{(2s-1)N^{2s-1}}$$

which is ≤ 1 provided

$$\log N \geq \frac{1}{2s-1}\log\frac{4}{2s-1}.$$

Let $s_+ = s_+(N)$ be the unique s such that the above holds with equality. Note that if $y = x\log x$ then in the limit as $x \to \infty$ we have $\log y \simeq \log x$ so that $x \simeq \frac{y}{\log y}$. It follows that, in the limit as $N \to \infty$ we have

$$\frac{1}{2s_+ - 1} \simeq \frac{\log N}{\log\log N}$$

so that applying Theorem 3.1 we now get

$$\text{H.dim}\, D_N \leq \frac{1}{2} + \frac{c\log\log N}{\log N}$$

for some constant $c > 0$.

Let $\sigma'_N(v)$ be the subset of $\sigma_N(v)$ consisting of those v' for which $\lfloor\frac{|v'|}{|v|}\rfloor \leq 2N$. For distinct $v', v'' \in \sigma'_N(v)$ with $v' = (p', q'), v'' = (p'', q'')$ we have

$$\left|\frac{p'}{q'} - \frac{p''}{q''}\right| \geq \frac{1}{q'q''} \geq \frac{1}{(2N+1)^2q^2}$$

whereas

$$\big|I(v')\big| \leq \frac{2}{N(q')^2} \leq \frac{2}{N^3q^2}$$

from which we see that the gap between $I(v')$ and $I(v'')$ is at least (assuming $N \geq 72$)

$$\frac{1}{9N^2q^2} - \frac{4}{N^3q^2} \geq \frac{1}{18N^2q^2} \geq \frac{1}{36N}\big|I_N(v)\big|.$$

Thus, (15) holds with $\rho = \frac{1}{36N}$. For any $0 < s < 1$ we have (using $N \geq 2$)

$$\begin{aligned}\sum_{N<a\leq 2N}\frac{2N^s}{(a+1)^{2s}(N+1)^s} &\geq \frac{1}{2s-1}\left(\frac{1}{(N+2)^{2s-1}} - \frac{1}{(2N+2)^{2s-1}}\right)\\ &\geq \frac{1}{3(2s-1)(N+2)^{2s-1}} \geq \frac{1}{6(2s-1)N^{2s-1}}\end{aligned}$$

which is ≥ 1 provided

$$\log N \leq \frac{1}{2s-1}\log\frac{1/6}{2s-1}.$$

Let $s_- = s_-(N)$ be the unique s such that the above holds with equality. It follows that in the limit as $N \to \infty$ we have

$$\frac{1}{2s_- - 1} \simeq \frac{\log N}{\log\log N}$$

so that applying Theorem 3.2 we now get

$$\mathrm{H.dim}\, D_N \geq \frac{1}{2} + \frac{c \log\log N}{\log N}$$

for some constant $c > 0$.

This establishes the following.

Theorem 3.4. *There are $c_2 > c_1 > 0$ such that for all $N \geq 72$*

$$\frac{1}{2} + \frac{c_1 \log\log N}{\log N} \leq \mathrm{H.dim}\, D_N \leq \frac{1}{2} + \frac{c_2 \log\log N}{\log N}.$$

Since $D_\infty = \cap_N D_N$, it follows that $\mathrm{H.dim}\, D_\infty \leq \frac{1}{2}$. To obtain the opposite inequality, one can repeat the argument for the lower bound on $\mathrm{H.dim}\, D_N$ with N "replaced" by a sequence N_k that slowly increases to infinity. The main issue is that the spacing condition (15) is no longer satisfied, and this is precisely the point where Theorem 3.3 is needed to complete the argument. We shall omit the details, since we can instead appeal to a classical result of Jarnik-Besicovitch. Let A_δ ($\delta > 0$) be the set of irrationals whose sequence of partial quotients satisfy $a_{k+1} > q_k^\delta$, where q_k is the height of the kth convergent. Then $A_\delta \subset D_\infty$ and the theorem of Jarnik-Besicovitch asserts that

$$\mathrm{H.dim}\, A_\delta \geq \frac{1}{2+\delta}.$$

Thus, we have established

Theorem 3.5. *The Hausdorff dimension of D_∞ is $\frac{1}{2}$.*

To the best of the author's knowledge, the results in this section have not appeared in the literature.

4. Upper bound calculation

In the rest of the paper, we assume $d = 2$.

Definition 4.1. For each $v \in Q$, let

$$\varepsilon(v)^{3/2} := \frac{|L(v)|}{|v|^{1/2}}$$

and define

$$Q_\varepsilon := \{v \in Q : \varepsilon(v) < \varepsilon\}.$$

Definition 4.2. For each $v \in Q$, let

$$\mathcal{L}^*(v) := \mathcal{L}(v) \setminus \{L(v)\}.$$

Fix, once and for all, an element $\hat{L} \in \mathcal{L}^*(v)$ such that $|\hat{L}|$ is minimal, and denote this element by

$$\hat{L}(v).$$

An important consequence of the assumption $d = 2$ is the following.

Lemma 4.3. *For any $v \in Q_\varepsilon$,*

$$\frac{|v|}{|L(v)|} \le |\hat{L}(v)| \le (1+\varepsilon^3)\frac{|v|}{|L(v)|} \tag{21}$$

and

$$|\hat{L}(v)| > \varepsilon^{-3/2}|v|^{1/2} \quad \textit{and} \quad |\hat{L}(v)| > \varepsilon^{-3}|L(v)|. \tag{22}$$

Proof. Let $L = L(v)$ and $\hat{L} = \hat{L}(v)$. We may think of them as vectors in the plane of lengths $|L|$ and $|\hat{L}|$, respectively, such that the area of the lattice they span is $|v|$. Without loss of generality we assume L is horizontal. The vertical component of $\hat{L}$ is then $\frac{|v|}{|L|}$ so that

$$\frac{|v|}{|L|} \le |\hat{L}| \le \frac{|v|}{|L|} + |L|$$

giving (21). If $|L| < \varepsilon^{3/2}|v|^{1/2}$ then the vertical component of $\hat{L}$ is greater than $\varepsilon^{-3/2}|v|^{1/2}$, giving the first inequality in (22). From this and $v \in Q_\varepsilon$, the second inequality in (22) follows. □

By Theorem 2.17, for any $\mathbf{x} \in \mathrm{Sing}^*(d)$ and any $\varepsilon > 0$ the elements $v \in \Sigma(\mathbf{x})$ belong to Q_ε if $|v|$ is large enough. Let $\mathcal{B}_\varepsilon = \{\Delta(v)\}_{v\in Q_\varepsilon}$ and define

$$\sigma_\varepsilon \subset Q_\varepsilon \times Q_\varepsilon$$

to consist of all pairs (v, v') such that $|v| < |v'|$ and v and v' realise a consecutive pair of local minima of the function $W_\mathbf{x}$ for some $\mathbf{x} \in \mathrm{Sing}^*(d)$. For each $\mathbf{x} \in \mathbb{R}^d$, we fix, once and for all, a sequence (v_j) in $\Sigma(\mathbf{x})$ such that each local minimum of $W_\mathbf{x}$ is realised by exactly one v_j, and, by an abuse of notation, we shall denote this sequence by the same symbol

$$\Sigma(\mathbf{x}).$$

Then it is easy to see that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \sigma_\varepsilon)$ is a self-similar covering of $\mathrm{Sing}^*(d)$ for any $\varepsilon > 0$: the map $\mathcal{E}$ can be realised by sending $\mathbf{x}$ to a tail of $\Sigma(\mathbf{x})$. However, Theorem 3.1 does not yield any upper bound because it happens that $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \sigma_\varepsilon) = \infty$ for all $\varepsilon > 0$, the main reason being that there is no way to bound the ratios

$$\frac{\operatorname{diam}\Delta(v')}{\operatorname{diam}\Delta(v)}$$

away from one.

4.1. **First acceleration.**

Definition 4.4. Suppose that $\Sigma(\mathbf{x}) = (v_j)$. We define

$$\hat{\Sigma}(\mathbf{x})$$

to be the subsequence of $\Sigma(\mathbf{x})$ consisting of those v_{j+1} such that

$$v_{j+1} \notin \mathbb{Z}v_j + \mathbb{Z}v_{j-1}.$$

Lemma 4.5. *The sequence $\hat{\Sigma}(\mathbf{x})$ has infinite length iff $\mathbf{x}$ does not lie on a rational line in $\mathbb{R}^d$.*

Proof. The sequence $\hat{\Sigma}(\mathbf{x})$ is finite iff there is a two-dimensional sublattice $L \subset \mathbb{Z}^{d+1}$ such that $v_j \in L$ for all large enough j. If $\mathbf{x}$ lies on the line $\ell \subset \mathbb{R}^d$ containing $\dot{v}$ for all $v \in L$ then the shortest vector in $g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}$ is realised by some vector in $g_t h_{\mathbf{x}} L$ for all large enough t. Conversely, $v_j \in L$ for all large enough j implies $\mathbf{x} = \lim \dot{v}_j \in \ell$. □

Lemma 4.6. *Let $\hat{\Sigma}(\mathbf{x}) = (u_k)$ where $\mathbf{x}$ does not lie on a rational line. Let $\delta > 0$ and $\varepsilon > 0$ be related by $\delta = \varepsilon^{1+1/d}$. Then $\mathbf{x} \in \mathrm{DI}_{\delta/2}(d)$ implies $\varepsilon(u_k) < \varepsilon$ for all sufficiently large k, which in turn implies $\mathbf{x} \in \mathrm{DI}_{2\delta}(d)$. In particular, $\mathbf{x} \in \mathrm{Sing}(d)$ iff $\varepsilon(u_k) \to 0$ as $k \to \infty$.*

Proof. Let $\Sigma(\mathbf{x}) = (v_j)$ and for each j, let $L_j = \mathbb{Z}v_j + \mathbb{Z}v_{j-1}$ and

$$\varepsilon_j = \frac{|L_j|}{|v_j|^{1-1/d}}.$$

Given u_k there are indices $i < j$ such that $u_k = v_i, \ldots, v_j = u_{k+1}$. Suppose that $j > i+1$. Then $v_{i+1} \in L_i$ so that $L_{i+1} \subset L_i$ and we have equality by Lemma 2.4. It follows by induction that $L_{j-1} = \cdots = L_i$, from which is follows that $\varepsilon_i > \cdots > \varepsilon_{j-1}$. Therefore, $\varepsilon(u_k) < \varepsilon$ for all sufficiently large k iff $\varepsilon(v_j) < \varepsilon$ for all sufficiently large j. The lemma now follows from Theorem 2.17. □

Definition 4.7. For any $u \in Q$, let

$$\mathcal{V}(u)$$

be the set of vectors in Q of the form $au + b\tilde{v}$ where a, b are relatively prime integers such that $|b| \leq a$ and $\tilde{v} \in Q$ is a vector that satisfies $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v}$, $|\tilde{v}| < |u|$ and $L(\tilde{v}) \neq L(u)$. The set

$$\mathcal{V}_\varepsilon(u)$$

is defined similarly, except that we additionally require $\tilde{v} \in Q_\varepsilon$ and $|\tilde{v}| < \varepsilon|u|$.

Lemma 4.8. *Let $u, u' \in \hat{\Sigma}(\mathbf{x})$ be consecutive elements with $|u| < |u'|$. Let $\tilde{v}, v \in \Sigma(\mathbf{x})$ be the elements that immediately precede u and u', respectively. Then $v \in \mathcal{V}(u)$, provided $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v} \neq L(\tilde{v})$.*

Proof. Suppose $\Sigma(\mathbf{x}) = (v_j)$ so that $u = v_i$, $v = v_j$, $\tilde{v} = v_{i-1}$ and $u' = v_{j+1}$ for some indices $i \leq j$. We shall argue by induction on $j \geq i$ to show that

(i) $v_j = au + b\tilde{v}$ for some integers $-a < b \leq a$, and
(ii) $v_j - v_{j-1} = a'u + b'\tilde{v}$ for some integers $-a' \leq b' \leq a'$.

This is clear if $j = i$. For $j > i$, since $v_j \in \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j-2}$, Lemma 2.4 implies that (v_j, v_{j-1}) is an integral basis for $\mathbb{Z}v_{j-1} + \mathbb{Z}v_{j-2}$. Since $|v_j| > |v_{j-1}| > |v_{j-2}|$, there is a positive integer c such that

$$v_j = cv_{j-1} + v_{j-2}, \quad \text{or} \quad v_j = cv_{j-1} + (v_{j-1} - v_{j-2}).$$

In either case $v_j = mu + n\tilde{v}$ where $(m, n) = (ca + a', cb + b')$ satisfies $-m < n \leq m$ by the induction hypothesis. Similarly, $v_j - v_{j-1} = m'u + n'\tilde{v}$ where $(m', n') = (ca - a', cb - b')$ satisfies $-m \leq n \leq m$, again, by the induction hypothesis. The lemma now follows from (i). $\square$

Definition 4.9. For any $\varepsilon > 0$, define

$$\hat{\sigma}_\varepsilon \subset Q_\varepsilon \times Q_\varepsilon$$

to be the set consisting of pairs (u, u') for which there exists $v \in \mathcal{V}_\varepsilon(u)$ such that $L(u') = \mathbb{Z}u' + \mathbb{Z}v \in \mathcal{L}^*(v)$.

Corollary 4.10. *Let $\delta = \varepsilon^{3/2}$ where $0 < \varepsilon < 1$. Then $(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat{\sigma}_\varepsilon)$ is a self-similar covering of $\mathrm{DI}_{\delta/2}(2)$.*

Proof. Let $\Sigma(\mathbf{x}) = (v_j)$ for a given $\mathbf{x} \in \mathrm{DI}_{\delta/2}(2)$. Lemma 4.6 implies for all large enough j we have

$$\|\mathbb{Z}v_j + \mathbb{Z}v_{j-1}\|_{\mathcal{L}(v_j)} = \frac{|v_{j-1} \wedge v_j|}{|v_j|^{1/2}} < \varepsilon^{3/2} < 1$$

so that $L(v_j) = \mathbb{Z}v_j + \mathbb{Z}v_{j-1}$ and $\varepsilon(v_j) < \varepsilon$. Lemma 4.8 now implies $\hat{\Sigma}(\mathbf{x})$ is eventually $\hat{\sigma}_\varepsilon$-admissible. $\square$

Our next task is to estimate $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat{\sigma}_\varepsilon)$. For this, we need to enumerate the elements of $\hat{\sigma}_\varepsilon(u)$ for any given $u \in Q_\varepsilon$.

Definition 4.11. For any $v \in Q$ and $L' \in \mathcal{L}^*(v)$, let

$$\mathcal{U}_\varepsilon(v, L')$$

be the set of $u' \in Q_\varepsilon$ such that $L' = L(u') = \mathbb{Z}u' + \mathbb{Z}v$.

Note that, by definition, for any $u' \in \hat{\sigma}_\varepsilon(u)$ there exists a $v \in \mathcal{V}_\varepsilon(u)$ and an $L' \in \mathcal{L}^*(v)$ such that $u' \in \mathcal{U}_\varepsilon(v, L')$. Hence, for any $f : Q_\varepsilon \to \mathbb{R}_+$

$$\sum_{u' \in \hat{\sigma}_\varepsilon(u)} f(u') \leq \sum_{v \in \mathcal{V}_\varepsilon(u)} \sum_{L' \in \mathcal{L}^*(v)} \sum_{u' \in \mathcal{U}_\varepsilon(v,L')} f(u').$$

Notation. We write $A \preceq B$ to mean $A \leq CB$ for some universal constant C. Note that $A \asymp B$ is equivalent to $A \preceq B$ and $B \preceq A$. We write $A \succeq B$ to mean the same thing as $B \preceq A$.

Proposition 4.12. *There is a constant C such that for any $u \in Q_\varepsilon$, and for any $s > 4/3$ and any $r < 6s - 3$*

$$\sum_{u' \in \hat{\sigma}_\varepsilon(u)} \left(\frac{\varepsilon(u)}{\varepsilon(u')}\right)^r \left(\frac{\operatorname{diam}\Delta(u')}{\operatorname{diam}\Delta(u)}\right)^s \leq \frac{C(6s-3-r)^{-1}\varepsilon^{6s-3-r}}{(3s-4)^2(\varepsilon(u))^{6-3s-r}}. \tag{23}$$

Proof. Given $v \in \mathcal{V}_\varepsilon(u)$ there are $a, b \in \mathbb{Z}$ with $|b| < a$ and $\tilde{v} \in Q_\varepsilon$ such that $v = au + b\tilde{v}$, $|\tilde{v}| < \varepsilon|u|$, $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v}$ and $L(u) \neq L(\tilde{v})$. Note that $\tilde{v} \in Q_\varepsilon$ implies $|\hat{L}(\tilde{v})| > \varepsilon^{-3/2}|\tilde{v}|^{1/2}$ and since $L(u) \neq L(\tilde{v})$, we have $|L(u)| \geq |\hat{L}(\tilde{v})|$, so that $u \in Q_\varepsilon$ now implies

$$|u| > \varepsilon^{-3}|L(u)|^2 > \varepsilon^{-6}|v|$$

so that

$$|v| \asymp a|u|.$$

Since $|\tilde{v}| < |u|$ and $(u, \tilde{v})$ is an integral basis for $L(u)$, there are at most two possibilities for $\tilde{v}$, so that given u and the positive integer a, there are at most $O(a)$ possibilities for v. It follows that for any $q > 2$

$$\sum_{v \in \mathcal{V}_\varepsilon(u)} \frac{|u|^q}{|v|^q} \preceq \sum_a \frac{1}{a^{q-1}} \asymp \frac{1}{q-2}. \tag{24}$$

Let $\mathcal{L}_+(v)$ be the set of elements in $\mathcal{L}(v)$ considered with orientations, and think of it as a subset of $\wedge^2\mathbb{Z}^3$. Note that addition is defined for those pairs $L, L' \in \mathcal{L}_+(v)$ whose $\mathbb{Z}$-span contains all of $\mathcal{L}_+(v)$. Let L and $\hat{L}$ be the elements in $\mathcal{L}_+(v)$ corresponding to a fixed choice of orientation for $L(v)$ and $\hat{L}(v)$, respectively. Each $L' \in \mathcal{L}^*(v)$ can be oriented so that, as an element in $\mathcal{L}_+(v)$ we have $L' = \tilde{a}\hat{L} + \tilde{b}L$ for some (relatively prime) integers $\tilde{a}, \tilde{b}$ with $\tilde{a} > 0$. Let

$$\hat{L}_m = \hat{L} + mL, \quad m \in \mathbb{Z}.$$

There is a unique integer m such that $L' = L_m$ or L' is a postive linear combination of $\hat{L}_m$ and $\hat{L}_{m+1}$. In any case, for each $L' \in \mathcal{L}^*(v)$ there is an integer m (and an orientation for L') such that

$$L' = a'\hat{L}_m + b'L \quad a' > b' \geq 0.$$

Note that $v \in Q_\varepsilon$ implies

$$|\hat{L}_m| \geq |\hat{L}| > \varepsilon^{-3/2}|v|^{1/2} > \varepsilon^{-3}|L|$$

so that

$$|L'| \asymp a'|\hat{L}_m|.$$

Let $N = \left\lfloor \frac{|\hat{L}|}{|L|} \right\rfloor$ so that $|\hat{L}_m| \asymp |L|(N+|m|)$ and

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{|\hat{L}|^p}{|\hat{L}_m|^p} &\asymp \frac{|\hat{L}|^p}{|L|^p} \sum_{m\geq 1} \frac{1}{(N+m)^p} \\
&\asymp \frac{|\hat{L}|^p}{|L|^p} \left(\sum_{m=1}^{N} \frac{1}{N^p} + \sum_{m>N} \frac{1}{m^p} \right) \\
&\asymp \frac{|\hat{L}|^p}{|L|^p} \left(\frac{1}{N^{p-1}} + \frac{1}{(p-1)N^{p-1}} \right) \asymp \frac{p}{p-1} \frac{|\hat{L}|}{|L|}.
\end{aligned}$$

Since there are at most $O(a')$ possibilities for L' given v, m and a', and since $|L||\hat{L}| \asymp |v|$, it follows that for any $q > 2$

$$\sum_{L'\in\mathcal{L}^*(v)} \frac{1}{|L'|^q} \preceq \sum_{m\in\mathbb{Z}} \frac{1}{|\hat{L}_m|^q} \sum_{a'} \frac{1}{(a')^{q-1}} \asymp \frac{1}{q-2} \frac{|L|^{q-2}}{|v|^{q-1}}. \tag{25}$$

Associate to each $u' \in \mathcal{U}_\varepsilon(v, L')$ the positive integer

$$c = \left\lceil \frac{|u'|}{|v|} \right\rceil > \frac{\varepsilon^{-3}|L'|^2}{|v|} \geq \varepsilon^{-3} \frac{|\hat{L}|^2}{|v|} \geq \varepsilon^{-3} \frac{|v|}{|L|^2} > \varepsilon^{-6}.$$

Since $|u'| \asymp c|v|$ and there are 2 possibilities for u', given v, L' and c, and since $c > \frac{\varepsilon^{-3}|L'|^2}{|v|}$, it follows that for any $p > 1$

$$\sum_{u'\in\mathcal{U}_\varepsilon(v,L')} \frac{1}{|u'|^p} \asymp \frac{1}{|v|^p} \sum_{c} \frac{1}{c^p} \asymp \frac{\varepsilon^{3p-3}}{(p-1)|v||L'|^{2p-2}}. \tag{26}$$

Using (26), (25) and (24) with $p = 2s - \frac{r}{3}$ and $q = 3s - 2$, we obtain

$$\sum_{v \in \mathcal{V}_\varepsilon(u)} \sum_{L' \in \mathcal{L}^*(v)} \sum_{u' \in \mathcal{U}_\varepsilon(v,L')} \left(\frac{|L'|}{|L|}\right)^{s-\frac{2r}{3}} \left(\frac{|u|}{|u'|}\right)^{2s-\frac{r}{3}}$$
$$\preceq \sum_{v \in \mathcal{V}_\varepsilon(u)} \sum_{L' \in \mathcal{L}^*(v)} \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|u|^{2s-\frac{r}{3}}}{|L|^{s-\frac{2r}{3}}|v||L'|^{3s-2}}$$
$$\preceq \sum_{v \in \mathcal{V}_\varepsilon(u)} \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|u|^{2s-\frac{r}{3}}|L|^{2s-4+\frac{2r}{3}}}{(3s-4)|v|^{3s-2}}$$
$$\preceq \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|L|^{2s-4+\frac{2r}{3}}}{(3s-4)^2|u|^{s-2+\frac{r}{3}}}$$

and this completes the proof of the proposition. $\square$

Proposition 4.12 implies there exists $C > 0$ such that for any $s > \frac{4}{3}$

$$\sum_{u' \in \hat{\sigma}_\varepsilon(u)} \left(\frac{\operatorname{diam} \Delta(u')}{\operatorname{diam} \Delta(u)}\right)^s \le \frac{C\varepsilon^{6s-3}}{(3s-4)^2(\varepsilon(u))^{6-3s}}. \tag{27}$$

However, this still does not imply $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat{\sigma}_\varepsilon) < \infty$ for any $\varepsilon > 0$.

4.2. **Second acceleration.** Let $\hat{\sigma}'_\varepsilon \subset Q \times Q$ be the set of all pairs (u, u') satisfying $u \in Q_\varepsilon$ and $u' \in \bigcup_{j \ge 1} \sigma''_j(u)$ where

$$\sigma''_1(u) := \hat{\sigma}_{\varepsilon(u)}(u), \qquad \sigma''_j(u) := \bigcup\{\hat{\sigma}_{\varepsilon(u)}(u') : u' \in \sigma'_{j-1}(u)\},$$
$$\sigma'_1(u) := \hat{\sigma}_\varepsilon(u), \qquad \sigma'_j(u) := \bigcup\{\hat{\sigma}_\varepsilon(u') : u' \in \sigma'_{j-1}(u)\}.$$

Proposition 4.13. $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat{\sigma}'_\varepsilon) = \frac{4}{3} + O(\varepsilon^{3/2})$.

Proof. To simplify notation, we denote the diameter of a set by $|\cdot|$. Given $s > \frac{4}{3}$, we apply Proposition 4.12 to ensure that if $\varepsilon > 0$ is sufficiently small then, by (27), for any $u \in Q_\varepsilon$

$$\sum_{u' \in \hat{\sigma}_\varepsilon(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \le \frac{1}{2}\left(\frac{\varepsilon}{\varepsilon(u)}\right)^{6-3s} \quad \text{and}$$
$$\sum_{u' \in \hat{\sigma}_\varepsilon(u)} \left(\frac{\varepsilon(u)}{\varepsilon(u')}\right)^{6-3s} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \le \frac{C\varepsilon^{9s-9}}{(3s-4)^2} \le \frac{1}{2}.$$

We may choose ε so that $\varepsilon^{9s-9} \asymp (3s-4)^2$; hence, $s = \frac{4}{3} + O(\varepsilon^{3/2})$.

For each $u' \in \sigma''_j(u)$ there are $u_1, \dots, u_{j-1} \in Q_\varepsilon$ such that

$$u_1 \in \hat{\sigma}_\varepsilon(u), \quad \dots, \quad u_{j-1} \in \hat{\sigma}_\varepsilon(u_{j-2}), \quad \text{and} \quad u' \in \hat{\sigma}_{\varepsilon(u)}(u_{j-1}).$$

$$\sum_{u'\in\sigma_j''(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \le \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-1}} \frac{|\Delta(u_{j-1})|^s}{|\Delta(u_{j-2})|^s} \sum_{u'} \frac{|\Delta(u')|^s}{|\Delta(u_{j-1})|^s}$$
$$\le \frac{1}{2} \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-1}} \frac{|\Delta(u_{j-1})|^s}{|\Delta(u_{j-2})|^s} \left(\frac{\varepsilon(u)}{\varepsilon(u_{j-1})}\right)^{6-3s}$$
$$\le \frac{1}{2^2} \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-2}} \frac{|\Delta(u_{j-2})|^s}{|\Delta(u_{j-3})|^s} \left(\frac{\varepsilon(u)}{\varepsilon(u_{j-2})}\right)^{6-3s}$$
$$\le \cdots \le \frac{1}{2^j}$$

so that

$$\sum_{u'\in\hat\sigma_\varepsilon'(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \le \sum_{j\ge1} \sum_{u'\in\sigma_j''(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \le \sum_{j\ge1} \frac{1}{2^j} \le 1$$

and therefore $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat\sigma_\varepsilon) \le s$. □

Now we verify that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat\sigma_\varepsilon')$ is a self-similar covering of $\mathrm{Sing}^*(2)$. Let $\mathcal{E}(\mathbf{x})$ be a subsequence (w_i) of $\hat\Sigma(\mathbf{x})$ such that $\varepsilon(w_i)$ is strictly decreasing to zero as $i \to \infty$, where the initial element w_0 is chosen so that for all v that occurs after w_0 in the sequence $\Sigma(\mathbf{x})$ we have $\varepsilon(v) < \varepsilon$. The sequence $\mathcal{E}(\mathbf{x})$ is $\hat\sigma_\varepsilon'$-admissible by construction. It follows that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat\sigma_\varepsilon')$ is a self-similar covering of $\mathrm{Sing}^*(2)$, and by Theorem 3.1 we can now conclude that

$$\mathrm{H.dim}\,\mathrm{Sing}^*(2) \le \frac{4}{3}.$$

We now describe how the preceding argument can be modified to give an upper bound estimate on $\mathrm{H.dim}\,\mathrm{DI}_\delta(2)$. Let $\mathrm{DI}_\delta^*(2)$ denote the set $\mathrm{DI}_\delta(2)$ with all rational lines removed. Extend the map $\mathcal{E}$ to all of $\mathrm{DI}_\delta^*(2)$ by choosing the subsequence (w_i) of $\hat\Sigma(\mathbf{x})$ so that $\varepsilon(w_i)$ is a monotone sequence, and such that $\varepsilon(w_0) < 2\lim_i \varepsilon(w_i)$. Modify the definition of $\hat\sigma_\varepsilon'$ by replacing the subscript $\varepsilon(u)$ in the formula for σ_j'', $j \ge 1$ with the expression $2\varepsilon(u)$. Then $(\mathcal{B}_\varepsilon, Q_\varepsilon, \hat\sigma_\varepsilon)$ is a self-similar covering of $\mathrm{DI}_\delta^*(2)$ provided $\delta = \frac{\varepsilon^{3/2}}{2}$. Theorem 3.1 and Proposition 4.13 now imply

$$\mathrm{H.dim}\,\mathrm{DI}_\delta(2) = \frac{4}{3} + O(\delta).$$

5. Lower bound calculation

Lemma 5.1. *Let $0 < \varepsilon < \frac{1}{2}$. Suppose $u \in Q$, $L' \in \mathcal{L}^*(u)$, and $u' \in L'$ is such that $L' = \mathbb{Z}u' + \mathbb{Z}u$ and $|u'| > \varepsilon^{-3}|L'|^2$. Then $L' = L(u')$,*

$u' \in Q_\varepsilon$, and

$$\overline{\Delta(u')} \subset \Delta(u).$$

Moreover, if $u \in Q_\varepsilon$ then $|u'| > \varepsilon^{-6}|u|$.

Proof. Since the norm of $L' \in \mathcal{L}(u')$ is

$$\|L'\|_{\mathcal{L}(u')} = \frac{|L'|}{|u'|^{1/2}} < \varepsilon^{3/2} < 1$$

we have $L' = L(u')$ and $\varepsilon(u') < \varepsilon$ so that $u' \in Q_\varepsilon$. Let $L = L(u)$ and note that since $L' \neq L$, we have

$$|L'| \geq |\hat{L}(u)| \geq \frac{|u|}{|L|}.$$

Then

$$\operatorname{dist}(\dot{u}, \dot{u}') = \frac{|u \wedge u'|}{|u||u'|} < \frac{\varepsilon^3}{|u||L'|} \leq \frac{\varepsilon^3 |L|}{|u|^2}.$$

The fact that the Euclidean length of the shortest nonzero vector in any two-dimensional unimodular lattice is universally bounded above by $\sqrt{2}$ implies that

$$\varepsilon(u)^3 < 2 \quad \text{for any} \quad u \in Q. \tag{28}$$

Therefore,

$$\frac{|L'||u|^2}{|L||u'|^2} \leq \frac{\varepsilon^6 |u|^2}{|L||L'|^3} \leq \frac{\varepsilon^6 |L|^2}{|u|} < 2\varepsilon^6$$

so that

$$\frac{\varepsilon|L|}{|u|^2} + \frac{2|L'|}{|u'|^2} < (\varepsilon^3 + 4\varepsilon^6)\frac{|L|}{|u|^2} < \frac{|L|}{2|u|^2}.$$

Theorem 2.10 now implies $\overline{\Delta(u')} \subset \Delta(u)$. If $u \in Q_\varepsilon$ then

$$|u'| > \varepsilon^{-3}|L'|^2 > \frac{\varepsilon^{-3}|u|^2}{|L|^2} > \varepsilon^{-6}|u|.$$

$\square$

Definition 5.2. For each $u \in Q$, let

$$\mathcal{N}_\varepsilon(u)$$

be the set of $u' \in Q$ such that $\mathbb{Z}u' + \mathbb{Z}u \in \mathcal{L}^*(u)$ and $|u'| > \varepsilon^{-3}|u \wedge u'|^2$.

Note that $\mathcal{N}_\varepsilon(u) \subset Q_\varepsilon$, by definition.

Theorem 5.3. *Let $0 < \varepsilon < \frac{1}{3}$. Suppose (u_k) is a sequence in Q satisfying $u_{k+1} \in \mathcal{N}_\varepsilon(u_k)$ for all $k \geq 0$. Then*

(a) *The limit $\mathbf{x} := \lim_k \dot{u}_k$ exists and $u_k \in \Sigma(\mathbf{x})$ for all k.*
(b) *$\mathbf{x} \in \mathrm{DI}_\delta(2)$, where $\delta = 2\varepsilon^{3/2}$.*

(c) *If* $\varepsilon(u_k) \to 0$ *as* $k \to \infty$ *then* $\mathbf{x} \in \mathrm{Sing}(2)$.

(d) *For all sufficiently large* t

$$W(t) - \log(1-\varepsilon^6) \le W_{\mathbf{x}}(t) \le W(t) \tag{29}$$

where

$$W(t) = \log \ell(g_t h_{\mathbf{x}}\{u_k\}) = \log \min_{k\ge 0} \|g_t h_{\mathbf{x}} u_k\|'.$$

Proof. Apply Lemma 5.1 with $L' = \mathbb{Z}u_{k+1} + \mathbb{Z}u_k$ to conclude that $\cap_k \Delta(u_k)$ is nonempty, and $u_k \in Q_\varepsilon$ for all $k \ge 1$. Moreover, $|u_k| \to \infty$ so that $\operatorname{diam}\Delta(u_k) \to 0$ and (a) follows. Lemma 5.1 also implies $L(u_{k+1}) = \mathbb{Z}u_{k+1} + \mathbb{Z}u_k$, so that, by Theorem 2.12, we have

$$\varepsilon_{\mathbf{x}}(u_k, u_{k+1})^{3/2} \le 2\frac{|u_k \wedge u_{k+1}|}{|u_{k+1}|^{1/2}} = 2\varepsilon(u_{k+1})^{3/2} < \delta.$$

Hence, Lemma 2.15 implies the local maxima of the piecewise linear function W are all bounded by $\log\delta$. Since $W_{\mathbf{x}} \le W$ (by monotonicity of ℓ), it follows that the local maxima of $W_{\mathbf{x}}$ are bounded by $\log\delta$, eventually. By Theorem 2.17, this means $\mathbf{x} \in \mathrm{DI}_\delta(2)$, giving (b). If $\varepsilon(u_k) \to 0$ then $W_{\mathbf{x}}(t) \to -\infty$, so that $\mathbf{x} \in \mathrm{Sing}(2)$. This proves (c).

Since $W_{\mathbf{x}} \le W$, the second inequality in (29) actually holds for all t. For the first inequality, we consider a local maximum time t for W. Then for some index k, if we set $u = g_t h_{\mathbf{x}} u_k$ and $u' = g_t h_{\mathbf{x}} u_{k+1}$ then $\|u\|' = \|u'\|'$. The corresponding local maximum value is $\log\varepsilon'$ where ε' is the common $\|\cdot\|'$-length of u and u'. To prove the first inequality in (29) it suffices to show that for any $w \in g_t h_{\mathbf{x}}\mathbb{Z}^3$ we have

$$\|w\|' \ge (1-\varepsilon^6)\varepsilon'. \tag{30}$$

Lemma 5.1 implies (for $k \ge 1$) $|u'| > \varepsilon^{-6}|u|$. Hence, $\|u' \pm u\|' \ge |u' \pm u| \ge (1-\varepsilon^6)\varepsilon'$. Note that $\|au' + bu\|' \ge \varepsilon'$ for any pair of integers with $|a| \ne |b|$. This establishes (30) for $w \in \mathbb{Z}u + \mathbb{Z}u'$. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{R}^3$. For any $v \in \mathbb{R}^3$ we have

$$\|v\|' \le \|v\|_e \le \sqrt{2}\|v\|'.$$

The Euclidean area of a fundamental parallelogram for $\mathbb{Z}u + \mathbb{Z}u'$ is

$$\|u \wedge u'\|_e \le \|u\|_e \|u'\|_e \le 2(\varepsilon')^2$$

so that for any $w \in g_t h_{\mathbf{x}}\mathbb{Z}^3 \setminus (\mathbb{Z}u + \mathbb{Z}u')$ we have

$$\|w\|' \ge \frac{\|w\|_e}{\sqrt{2}} \ge \frac{1}{2\sqrt{2}(\varepsilon')^2}$$

which is $> \varepsilon'$ since

$$\varepsilon' \le 3^{2/3}\varepsilon(u_{k+1}) < 3^{2/3}\varepsilon < \frac{1}{\sqrt[3]{3}} < \frac{1}{\sqrt{2}}.$$

Thus (30) holds for all $w \in g_t h_{\mathbf{x}} \mathbb{Z}^3$. $\square$

We assume for each $u \in Q$ orientations for $L(u)$ and $\hat{L}(u)$ have been chosen so that we may think of them as elements of $\wedge^2 \mathbb{Z}^3$.

Definition 5.4. Given $u \in Q$ and integers $a \geq b \geq 0$ such that $\gcd(a,b) = 1$ we set

$$L' = a'\hat{L}(u) + b'L(u).$$

Additionally, given $0 < \varepsilon < 1$ and an integer $c \geq 1$ satisfying

$$M_\varepsilon < c < 2M_\varepsilon - 1 \quad \text{where} \quad M_\varepsilon = \frac{\varepsilon^{-3}|L'|^2}{|u|}$$

we define $\psi_\varepsilon(u,a,b,c)$ to be the unique $u' \in Q$ such that

$$L' = u' \wedge u \quad \text{and} \quad \left\lfloor \frac{|u'|}{|u|} \right\rfloor = c.$$

Note that $\psi_\varepsilon(u,a,b,c) \in \mathcal{N}_\varepsilon(u)$ because $L' = L(u')$. Note also that $c > M_\varepsilon$ implies $\psi_\varepsilon(u,a,b,c) \in Q_\varepsilon$ while $c < 2M_\varepsilon - 1$ implies $\psi_\varepsilon(u,a,b,c) \notin Q_{\varepsilon/2}$. Therefore, we always have

$$\psi_\varepsilon(u,a,b,c) \in \mathcal{N}_\varepsilon(u) \cap Q'_\varepsilon \quad \text{where} \quad Q'_\varepsilon := Q_\varepsilon \setminus Q_{\varepsilon/2}.$$

Lemma 5.5. *Let $0 < \varepsilon < 2^{-7}$. If $u' = \psi_\varepsilon(u,a,b,c)$ and $u'' = \psi_\varepsilon(u,a',b',c')$ are such that $(a,b) \neq (a',b')$ or $|c - c'| \geq 20$ then*

$$\operatorname{dist}(\Delta(u'),\Delta(u'')) \geq \frac{\varepsilon^9}{2^{11}N^3} \operatorname{diam} \Delta(u) \tag{31}$$

where $N = \max(a,a')$.

Proof. Let $L = L(u)$, $\hat{L} = \hat{L}(u)$ and $L' = u' \wedge u$. Note that by (21) we have

$$|L'| \leq 2a|\hat{L}| \leq \frac{4N|u|}{|L|}.$$

Theorem 2.10 implies

$$\operatorname{diam} \Delta(u) \leq \frac{4|L|}{|u|^2}$$

and also $\Delta(u') \subset B(\dot{u}', 2r')$ where

$$r' = \frac{|L'|}{|u'|^2} \leq \frac{|L'|}{c^2|u|^2} < \frac{|L'|}{M_\varepsilon^2|u|^2} = \frac{\varepsilon^6}{|L'|^3}.$$

If $(a,b) = (a',b')$ then

$$\operatorname{dist}(\dot{u}',\dot{u}'') = \frac{|u' \wedge u''|}{|u'||u''|} > \frac{20|L'|}{(c+1)^2|u|^2} > \frac{5|L'|}{M_\varepsilon^2|u|^2} = \frac{5\varepsilon^6}{|L'|^3}$$

so that
$$\mathrm{dist}(\Delta(u'),\Delta(u'')) \geq \frac{\varepsilon^6}{|L'|^3} \geq \frac{\varepsilon^6|L|^3}{2^6N^3|u|^3} \geq \frac{\varepsilon^9|L|}{2^9N^3|u|^2}$$
giving (31) in the case $(a,b)=(a',b')$. If $(a,b)\neq(a',b')$, let $L''=u''\wedge u$ and note that
$$\sin\angle\pi_u(L')\pi_u(L'') = \frac{|u|}{|L'||L''|} \geq \frac{|L|^2}{16N^2|u|} \geq \frac{\varepsilon^3}{2^7N^2}$$
and
$$\mathrm{dist}(\dot u,\dot u') = \frac{|u\wedge u'|}{|u||u'|} \geq \frac{|L'|}{2M_\varepsilon|u|^2} = \frac{\varepsilon^3}{2|u||L'|} \geq \frac{\varepsilon^3|L|}{8N|u|^2}$$
so that
$$\mathrm{dist}(\dot u',\dot u'') \geq \frac{\varepsilon^6|L|}{2^{10}N^3|u|^2}.$$
Considering the component of L' perpendicular to L, as in the proof of Lemma 4.3, we get
$$|L'| \geq \frac{a|u|}{|L|} \geq \frac{N|u|}{|L|}$$
so that
$$2r' < \frac{2\varepsilon^6}{|L'|^3} \leq \frac{2^7\varepsilon^6|L|^3}{N^3|u|^3} < \frac{2^8\varepsilon^6|L|}{|u|^2}$$
by (28). Since $\varepsilon<2^{-7}$, it follows that
$$\mathrm{dist}(\Delta(u'),\Delta(u'')) \geq (\frac{1}{2^{10}} - 2^9\varepsilon^3)\frac{\varepsilon^6|L|}{N^3|u|^2} \geq \frac{\varepsilon^6}{2^{13}N^3}\,\mathrm{diam}\,\Delta(u)$$
which easily implies (31). □

The next proposition completes the proof of Theorem 1.3.

Proposition 5.6. *There is a constant $c>0$ such that for $0<\delta<2^{-10}$*
$$\mathrm{H.dim}\,\mathrm{DI}_\delta(2) \geq \frac{4}{3} + exp(-c\delta^{-4}).$$

Proof. Fix a parameter N to be determined later and set
$$\sigma_\varepsilon(u) = \{\psi_\varepsilon(u,a,b,c) : a\leq N,\ 20|c\}.$$
Fix $u_0\in Q$, let $U_0=\{u_0\}$ and recursively define
$$U_{k+1} = \bigcup_{u\in U_k}\sigma_\varepsilon(u)$$
where ε is defined by $\delta=3\varepsilon^{3/2}$. Note that
$$E_k = \bigcup_{u\in U_k}\overline{\Delta(u)}$$

is a disjoint union, by Lemma 5.5. We have $E_{k+1} \subset E_k$ by Lemma 5.1, and by Theorem 5.3(a), there is a one-to-one correspondence between the points of $E = \cap E_k$ and the sequences (u_k) starting with u_0 and satisfying $u_{k+1} \in \sigma_\varepsilon(u_k)$ for all k. Theorem 5.3(b) implies $E \subset \mathrm{DI}_\delta(2)$. The hypotheses (i)-(iii) of Theorem 3.2 now hold with

$$\rho = \frac{\varepsilon^9}{2^{11} N^3}.$$

Before checking (iv), we note that given $1 \leq a \leq N$ we have $\phi(a)$ choices for b such that $a \geq b \geq 0$ and $\gcd(a,b) = 1$, where ϕ is the Euler totient function. It is well known that

$$\liminf_{n\to\infty} \frac{\phi(n) \log\log n}{n} > 0.$$

Now, for (iv) we compute (assuming $s > \frac{4}{3}$)

$$\begin{aligned}
\sum_{u' \in \sigma_\varepsilon(u)} \frac{|L'|^s |u|^{2s}}{|L|^s |u'|^{2s}} &\asymp \sum_{L'} \frac{|L'|^s}{|L|^s} \sum_c \frac{1}{c^{2s}} \\
&\asymp \sum_{L'} \frac{|L'|^s}{|L|^s} \left(\frac{\varepsilon^3 |u|}{|L'|^2} \right)^{2s-1} \\
&\asymp \frac{\varepsilon^{6s-3} |u|^{2s-1}}{|L|^s |\hat{L}|^{3s-2}} \sum_{a'} \frac{\phi(a')}{(a')^{3s-2}} \\
&\succeq \varepsilon^{9s-6} \int_e^N \frac{dx}{x^{3s-3} \log x}.
\end{aligned}$$

Note that as $p \to 1^+$

$$\begin{aligned}
\int_e^\infty \frac{dx}{x^p \log x} &\asymp \sum_{k\geq 1} \int_{e^k}^{e^{k+1}} \frac{dx}{x^p k} = \sum_{k \geq 1} \frac{e^{-k(p-1)}}{(p-1)k} (1 - e^{-(p-1)}) \\
&= \frac{1 - e^{-(p-1)}}{p-1} \log \frac{1}{1 - e^{-(p-1)}} \asymp \log \frac{1}{p-1}.
\end{aligned}$$

Thus, we conclude that there is a constant $C > 1$ such that for any $s > \frac{4}{3}$ satisfying

$$\varepsilon^{9s-6} \left| \log \left(s - \frac{4}{3} \right) \right| > C$$

the condition (iv) of Theorem 3.2 holds by choosing N large enough (depending on ε). Since $\delta = 3\varepsilon^{3/2}$, the proposition follows. □

We now describe how to modify the preceding argument to obtain the lower bound in Theorem 1.1. Fix a parameter $C > 1$ to be determined

later and choose positive sequences $\varepsilon_k \to 0$ and $N_k \to \infty$ such that for all k,

$$\varepsilon_k < 2^{-7}, N_k \geq 1, \quad \text{and} \quad \varepsilon_k^6 \log\log N_k > C.$$

We shall also assume the sequences are slowly varying in the sense that the ratio of consecutive terms are bounded above and below by positive constants, say 2 and $\frac{1}{2}$. For example,

$$N_k = k+1, \qquad \varepsilon_k = \frac{1}{2^7 \log\log\log(k+C')}$$

where $C' > 1$ is chosen large enough depending only on C. The definition of the sets U_k are modified by the formula

$$U_{k+1} = \bigcup_{u \in U_k} \sigma_{\varepsilon_{k+1}}(u).$$

With E defined the same way as before, Theorem 5.3(c) now implies $E \subset \mathrm{Sing}(2)$. For each $u \in U_k$ set

$$\rho(u) = \frac{\varepsilon_{k+1}^9}{2^{11} N_k^3}$$

so that (i)-(iii) of Theorem 3.3 hold. The main calculation in the proof of Proposition 5.6 with $s = \frac{4}{3}$ now yields

$$\sum_{u' \in \sigma_{\varepsilon_{k+1}}(u)} \frac{|L'|^s |u|^{2s}}{|L|^s |u'|^{2s}} \succeq \varepsilon_{k+1}^6 \log\log N_k$$

so that (iv) of Theorem 3.3 holds provided C was chosen large enough at the beginning. It follows that

$$\mathrm{H.dim}\,\mathrm{Sing}(2) \geq \frac{4}{3}$$

and this completes the proof of Theorem 1.1.

6. Slowly divergent trajectories

In this section, we prove

Theorem 6.1. *Given any function $W(t) \to -\infty$ as $t \to \infty$ there exists a dense set of $\mathbf{x} \in \mathrm{Sing}^*(2)$ with the property $W_{\mathbf{x}}(t) \geq W(t)$ for all sufficiently large t.*

This answers affirmatively a question of A.N. Starkov [21] concerning the existence of slowly divergent trajectories for the flow on $\mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ induced by g_t.

Lemma 6.2. *Given $\delta > 0$ and a function $F(t) \to \infty$ as $t \to \infty$ there exists $t_0 > 0$ and a monotone function $f(t) \to \infty$ as $t \to \infty$ such that*

(i) $f(t) \leq F(t)$ *for all* $t > t_0$, *and*
(ii) $f(t + f(t)) \leq f(t) + \delta$ *for all* $t > t_0$.

Proof. We may reduce to the case where $F(t)$ is a nondecreasing function. Let t_0 be large enough so that $y_0 = F(t_0) > 0$ and for $k > 0$ set $t_k = t_{k-1} + y_{k-1}$ and $y_k = \min(F(t_k), y_{k-1} + \delta)$. Since $y_k \geq y_0 > 0 \forall k$ we have $t_k \to \infty$ and therefore also $y_k \to \infty$. Let $f(t) = y_k$ for $t_k \leq t < t_{k+1}$ so that $f(t) = y_k \leq F(t_k) \leq F(t)$ since $t_k \leq t$ and $F(t)$ is nondecreasing. Moreover, $t_{k+1} = t_k + y_k \leq t + f(t) < t_{k+1} + y_k \leq t_{k+1} + y_{k+1} = t_{k+2}$ so that $f(t + f(t)) = y_{k+1} \leq y_k + \delta = f(t) + \delta$. □

Definition 6.3. For any $v \in Q$, let

$$\tau(v) := -\frac{1}{3} \log \frac{|L(v)|}{|v|^2} = -\frac{1}{2} \log \frac{\varepsilon(v)}{|v|}.$$

Lemma 6.4. *There exists* $C > 0$ *such that for any* $0 < \varepsilon' < 1$ *and any* $u \in Q$, *there exists* $u' \in \mathcal{N}_{\varepsilon'}(u)$ *such that*
(i) $|\log \varepsilon(u') - \log \varepsilon'| \leq C$, *and*
(ii) $\big|\tau(u') - \tau(u) - 2|\log \varepsilon'| - |\log \varepsilon(u)|\big| \leq C$.

Proof. Let $L' = \hat{L}(u)$ and let $u' \in Q$ be determined by $L' = u \wedge u'$ and

$$|u'| > (\varepsilon')^{-3}|L'|^2 \geq |u'| - |u|.$$

Then $u' \in \mathcal{N}_{\varepsilon'}(u)$ by the first inequality. By Lemma 4.3 and (28)

$$|u'| > |L'|^2 \geq \frac{|u|^2}{|L(u)|^2} > 2|u|$$

so that $\varepsilon(u')^3 \asymp \frac{|L'|^2}{|u'|} \asymp (\varepsilon')^3$, giving (i). Since

$$\begin{aligned} \tau(u') - \tau(u) &= \frac{1}{2} \log \frac{|u'|}{|u|} - \frac{1}{2} \log \frac{\varepsilon'}{\varepsilon(u)} + O(1) \\ &= 2|\log \varepsilon'| + |\log \varepsilon(u)| + O(1) \end{aligned}$$

(ii) follows. □

Proof of Theorem 6.1. Let $\tilde{f}$ be the function obtained by applying Lemma 6.2 with $F = -W(t)$ and some given $\delta > 0$ to be determined later. Set $f = 3^{-1}\tilde{f}$ and note that f satisfies
(i) $3f(t) \leq -W(t)$ for all $t > t_0$, and
(ii) $f(t + 3f(t)) \leq f(t) + \delta$ for all $t > t_0$

and since $f(t) \to \infty$, given any $A > 0$ we can choose t_0, perhaps even larger, so that, in addition to (i) and (ii), f also satisfies
(iii) $f(t + 3f(t) + A) \leq f(t) + 2\delta$ for all $t > t_0$.

We claim there is a constant B such that for any $u \in Q_1$ satisfying

$$|f(\tau(u)) + \log \varepsilon(u)| \le B \tag{32}$$

and such that $|u|$ larger than some constant depending only on f there exists $u' \in \hat{\sigma}_1(u)$ such that

$$|f(\tau(u')) + \log \varepsilon(u')| \le B.$$

Indeed, given u satisfying (32), we let u' be obtained by applying Lemma 6.4 with $\varepsilon' < 1$ determined by

$$|\log \varepsilon'| = f\big(\tau(u) + |\log \varepsilon(u)|\big).$$

Then, if $A \ge 3B$ we have

$$\begin{aligned}|\log \varepsilon'| &\le f(\tau(u)) + 3f(\tau(u)) + 3B \\ &\le f(\tau(u)) + 2\delta \\ &\le |\log \varepsilon(u)| + B + 2\delta.\end{aligned}$$

By Lemma 6.4,

$$\begin{aligned}\tau(u') &\le \tau(u) + 2|\log \varepsilon'| + |\log \varepsilon(u)| + C \\ &\le \tau(u) + 3|\log \varepsilon(u')| + 2B + C + 4\delta \\ &\le \tau(u) + 3f(\tau(u)) + 5B + C + 4\delta\end{aligned}$$

so that if $A \ge 5B + C + 4\delta$ we have

$$\begin{aligned}f(\tau(u')) &\le f(\tau(u)) + 2\delta \\ &\le |\log \varepsilon'| + 2\delta \\ &\le |\log \varepsilon(u')| + C + 2\delta.\end{aligned}$$

Now,

$$|\log \varepsilon'| \ge f(\tau(u)) \ge |\log \varepsilon(u)| - B$$

so that

$$\tau(u') \ge \tau(u) + 3|\log \varepsilon(u)| - 2B - C.$$

Assuming $|u|$ large enough so that $3|\log \varepsilon(u)| \ge 2B + C$ we have

$$\begin{aligned}f(\tau(u')) &\ge f(\tau(u)) \\ &\ge f\big(\tau(u) + 3f(\tau(u)) + A\big) - 2\delta \\ &\ge f\big(\tau(u) + 3|\log \varepsilon(u)| + A - 3B\big) - 2\delta \\ &\ge |\log \varepsilon'| - 2\delta \\ &\ge |\log \varepsilon(u')| - C - 2\delta.\end{aligned}$$

Setting $A = 6C + 14\delta$, we see that the claim follows with $B = C + 2\delta$.

Given any nonempty open set $U \subset \mathbb{R}^2$, we can choose $u_0 \in Q$ such that $\Delta(u_0) \subset U$. Indeed, choose any $\mathbf{x}_0 \in U \setminus \mathbb{Q}^2$ and let $u_0 \in \Sigma(\mathbf{x}_0)$ be

such that $|u_0|$ is large enough so that $\Delta(u_0) \subset U$. Let δ be chosen large enough at the beginning so that (32) holds for $u = u_0$. Let $\Sigma_0 = (u_k)$ be a sequence constructed by recursive definition using the claim. Since

$$\tau(u_{k+1}) = \tau(u_k) + 3|\log \varepsilon(u_k)| + O(1) \tag{33}$$

and $\varepsilon(u_k) \asymp \exp(-f(\tau(u_k)))$ by construction, by choosing $|u_0|$ large enough initially we can ensure that $\varepsilon(u_k) < \frac{1}{3}$ for all k so that $\tau(u_k)$ increases to infinity as $k \to \infty$. Since $f(t) \to \infty$ as $t \to \infty$, this implies $\varepsilon(u_k) \to 0$ as $k \to \infty$. By construction, $u_{k+1} \in \mathcal{N}_{\varepsilon_k}(u_k)$ so that Theorem 5.3(c) implies

$$\mathbf{x} := \lim_k \dot{u}_k \in \mathrm{Sing}(2).$$

If $\mathbf{x}$ lies on a rational line, then $W_{\mathbf{x}}(t) \leq -\frac{t}{2} + C$ for some constant C and all large enough t. It is clear that we could have, at the start, reduced to the case where, say, $W(t) > -\log t$ for all t, so that $\mathbf{x} \in \mathrm{Sing}^*(2)$.

Let $D = |\log(1-3^{-6})|$. Theorem 5.3(d) implies for all $t \in [\tau(u_k), \tau(u_{k+1})]$

$$\begin{aligned} -W_{\mathbf{x}}(t) &\leq 3|\log \varepsilon(u_k)| + D \\ &\leq 3f(\tau(u_k)) + 3B + D \\ &\leq -W(\tau(u_k)) + 3B + D \\ &\leq -W(t) + 3B + D. \end{aligned}$$

It is clear that we could have chosen f initially to satisfy (i') $3f(t) \leq -W(t) - 3B - D$ for all $t > t_0$ instead of (i). With this choice, we conclude $W_{\mathbf{x}}(t) \geq W(t)$ for all $t > t_0$. □

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